

# Unit - 4 Algebraic Structures

And

## Morphism

\* Task: 1 Binary Composition and its properties.

1 Define binary Operation with Example.

Let  $A$  is a non-empty set, then  $a \times a$

$A \times A = \{(a, b) \mid a, b \in A\}$  then function

$* A \times A \rightarrow A$  is said binary operation  $A$ .

IF  $* (a, b) = a * b \in A$ .

EX

- + is binary Operation on  $N, Z, R$ .
- is binary Operation on  $N, Z, R$ .
- is not binary Operation on  $N$ .

2 Write all the properties of binary operations with Examples.

Let  $a, b, c$  are non empty set then, they follow this properties.

(1) Closure:  $a, b$  is close under the binary Operation.

$$\text{EX. } a, b \in G \Rightarrow a * b \in G$$

(2) Associative: For all  $a, b, c \in G$

$$a * (b * c) = (a * b) * c$$

(3) Existence of Identity: They existies of element,

$$e \in G, \text{ Such that,}$$

$$\text{For } \forall a, b, c \in G$$

$$a * e = a = e * a$$

(4) Existence of Inverse: For every  $b \in G$ , they every  $e \in G$

$$\forall a, b \in G$$

$$a * b = e = \text{Identity.}$$

Ex.  $(\mathbb{Z}, *)$ ,  $\forall a, b \in \mathbb{Z}$ ,  $a * b = a + b + 5$ .

Let  $a, b$  is non-empty set on  $\mathbb{Z}$  then,

(1) Closure:

For  $\forall a, b \in \mathbb{Z}$ ,  $a * b = a + b + 5 \in \mathbb{Z}$

So,  $\langle \mathbb{Z}, * \rangle$  is closure under the binary operation.

(2) Associative:

For  $\forall a, b, c \in \mathbb{Z}$ ,

$$(a * b) * c = (a + b + 5) * c \\ = a + b + c + 10 \quad \text{--- (1)}$$

$$(a) * (b * c) = a * (b + c + 5) \\ = a + b + c + 10 \quad \text{--- (2)}$$

From eq<sup>n</sup>, 1 = 2  $\langle \mathbb{Z}, * \rangle$  is Associative under the operation.

(3) Existence of Identity:

For  $\forall a, b, c \in \mathbb{Z}$  and  $e \in \mathbb{Z}$

$$\therefore a * e = a$$

$$\therefore a + e + 5 = a$$

$$\therefore e = -5 \in \mathbb{Z}$$

So,  $\langle \mathbb{Z}, * \rangle$  has identity element  $-5$ .

(4) Existence of Inverse:

$$\forall a, b \in \mathbb{Z}$$

$$b = a^{-1}$$

$$a * b = e$$

$$\therefore a + b + 5 = -5$$

$$\therefore a + b = -10 \in \mathbb{Z}$$

$$\therefore a^{-1} = -10 - a \in \mathbb{Z}$$

So,  $\langle \mathbb{Z}, * \rangle$  follows all the properties.

3 Test the following Properties for the given structure.

Properties: Closure, Identity, Inverse.

(a)  $(\mathbb{Z}, *)$ ,  $\forall a, b \in \mathbb{Z}$ ,  $a * b = a + b + 5$

Let  $\mathbb{Z}$  is not set empty set  
and  $a, b \in \mathbb{Z}$

(1) Closure:

$$\text{For } \forall a, b \in \mathbb{Z}, a * b = a + b + 5 \in \mathbb{Z}$$

So,  $\langle \mathbb{Z}, * \rangle$  is follow closure properties.

(2) Identity:

$$\text{For } \forall a, e \in \mathbb{Z},$$

$$a * e = a$$

$$\therefore a + e + 5 = a$$

$$\therefore e = -5 \in \mathbb{Z}$$

For  $\langle \mathbb{Z}, * \rangle$ , identity element is  $-5$ .

(3) Inverse:

$$\text{For } \forall a, b \in \mathbb{Z}$$

$$\therefore a * b = e$$

$$\therefore a + b + 5 = -5$$

$$\therefore a = -10 + b^{-1} \in \mathbb{Z}$$

For  $\langle \mathbb{Z}, * \rangle$ , Inverse property is satisfy.

(b)  $(Z^+, *)$ ,  $\forall a, b \in Z^+$ ,  $a * b = a^b$

Let,  $Z^+$  is not empty set and  
 $a, b \in Z^+$ .

(1) Closure:

For  $\forall a, b \in Z^+$ ,  $a * b = a^b \in Z^+$

(2) Existence of Identity:

For  $\forall a, b \in Z^+$  and  $e \in Z^+$

$$\therefore a * e = a$$

$$\therefore a^e = a$$

$$\therefore e = 1 \in Z^+$$

(3) Existence of Inverse:

For  $\forall a, b \in Z^+$ ,

$$a * a^{-1} = e$$

$$a^{a^{-1}} = 1$$

$$\therefore a \cdot \log a^{-1} = 1$$

$$\log a^{-1} = \frac{1}{a}$$

(4) Associative:

For,  $a, b, c \in \mathbb{Z}^+$

$$\begin{aligned} (a * b) * c &= a^b * c \\ &= (a^b)^c \in \mathbb{Z}^+ \quad - (1) \end{aligned}$$

$$\begin{aligned} a * (b * c) &= a * b^c \\ &= a^{b^c} \in \mathbb{Z}^+ \quad - (2) \end{aligned}$$

(c)  $(R, *)$ ,  $\forall a, b \in R$ ,  $a * b = \frac{a}{a+b}$

Let  $R$  is non empty set and  $a, b \in R$ .

(1) Closure:

For  $\forall a, b \in R$ ,

$$a * b = \frac{a}{a+b} \in R.$$

(2) Associative:

For,  $a, b, c \in \mathbb{Z}^+$

$$(a * b) * c = \frac{a}{a+b} * c$$

$$= \frac{a}{a+b+c} \quad \text{--- (1)}$$

$$a * (b * c) = a * \frac{b}{b+c}$$

$$= \frac{a}{b/(b+c)}$$

$$= \frac{a(b+c)}{b} \quad \text{--- (2)}$$

From eq<sup>n</sup> 1  $\neq$  2  
 $\therefore \langle R, * \rangle$  not follow Associative

(3) Existence of Identity:

For,  $a, b \in R, e \in R$

$$\therefore a * e = a$$

$$\therefore \frac{a}{a+e} = a$$

$$\therefore a+e = 1$$

$$\therefore e = 1-a \in R$$

For  $\langle R, * \rangle$  identity element is  $1-a$ .



(4) Existence of Inverse:

Let,  $a, a^{-1} \in R$

$$\therefore a * a^{-1} = e$$

$$\therefore \frac{a}{a + a^{-1}} = 1 - a$$

$$\therefore \frac{a}{1 - a} - a = a^{-1}$$

$$\therefore a^{-1} = \frac{a}{1 - a} - a \in R$$

For  $\langle R, * \rangle$ , Inverse of  $a$  is  $\frac{a}{1 - a} - a$

## \* Task 2 : Various Algebraic Structures and Order of element.

1 Define the following terms.

(a) Semi Group:

Let,  $G$  is non empty set and  $(G, *)$  is composition  $*$  is follow Associative then,  $G$  is said to be Semi Group.

$$(a * b) * c = a * (b * c)$$

(b) Monoid:

Let,  $G$  is non empty set and  $(G, *)$  is existes an identity element then,  $G$  is said to be Monoid.

$$\therefore a * e = a$$

(c) Group:

Let,  $G$  is non empty set and  $(G, *)$  is follow this properties, then  $G$  is said to be Group.

For  $a, b, c \in G$

- ures
- 1) Closure:  $a + b \in G$
  - 2) Associative:  $a * (b * c) = (a * b) * c$
  - 3) Identity:  $a * e = a$
  - 4) Inverse:  $a * a^{-1} = e$

### (d) Abelian Group:

A Group  $(G, *)$  is said to be Abelian Group,

IF,  $\forall a, b \in G, a * b = b * a$

$G$  is follow the Commulative Group Property.

### (e) Order of the Group:

Let  $(G, *)$  be a Group, then number of element in  $G$  is said to be order of Group.

(f) Order of element: The order of an element  $g$  in A Group  $G$  is the smallest positive integer such that,

$$g^n = e$$

The Order of an element  $G$  is denoted by  $O(g)$ .

2 Prove that set of positive rational number forms an Abelian Group under the composition of

$$a * b = \frac{ab}{2}$$

$\Rightarrow$  For, abelian Group, Commulative Group Property must be follow.

$$\therefore a * b = \frac{ab}{2} \in R \quad - (1)$$

$$\therefore b * a = \frac{ba}{2} = \frac{ab}{2} \in R \quad - (2)$$

From, eq<sup>n</sup> 1 = 2

So, Set of Rational  $\mathbb{Q}$  number is said to Abelian Group.

3 Check whether  $(R, \times)$  is Group or not.

Let,  $R$  is non empty set and  $a, b, c \in R$ .

For be a Group,  $R$  is must follow this four Properties.

## 1 Closure:

For  $a, b \in R$ ,

$$\therefore a + b \in R$$

So,  $R$  satisfy closure Properties.

## 2 Associative:

For  $a, b, c \in R$ ,

$$\begin{aligned} (a * b) * c &= a \cdot b * c \\ &= abc - \textcircled{1} \end{aligned}$$

$$\begin{aligned} a * (b * c) &= a * bc \\ &= abc - \textcircled{2} \end{aligned}$$

From eq<sup>n</sup> 1 = 2.

So,  $R$  is satisfy Associative Property.

## 3 Existance of Group Identity:

For  $a, b, c \in R$

$$a * e = a$$

$$\therefore ae = a$$

$$\therefore e = 1$$

For  $R$ , Identity element is 1.

#### 4 Existence of Inverse:

For  $a, e \in R$

$$\therefore a * a^{-1} = e$$

$$\therefore aa^{-1} = 1$$

$$\therefore a^{-1} = \frac{1}{a}$$

Here,  $R$  is satisfy the all the Properties.

Hence,  $R$  is a Group.

#### 4 Classify the algebraic structure for,

(a)  $(Z_7 - \{0\}, X_7)$

$$Z_7 = \{[1], [2], [3], [4], [5], [6]\}$$

(i) Closure: From Composition table, For every  $[i][j] \in Z_7$ .

(ii) Associative: From Composition table, it is clear that  $Z_7$  is associative.

→ Composition Group:

$X_7$	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]
[2]	[2]	[4]	[6]	[1]	[3]	[5]
[3]	[3]	[6]	[2]	[5]	[1]	[4]
[4]	[4]	[1]	[5]	[2]	[6]	[3]
[5]	[5]	[3]	[1]	[6]	[4]	[2]
[6]	[6]	[5]	[4]	[3]	[2]	[1]

(iii) Identity: From Composition table, identity elements is [1].

(iv) Inverse: From Composition table,

Inverse of [1] is [1].

Inverse of [2] is [4].

Inverse of [3] is [5].

Inverse of [4] is [2].

Inverse of [5] is [3].

Inverse of [6] is [6].

Hence,  $(Z_7 - \{0\}, X_7)$  is algebraic structure.

(2)  $(M_{22}, X)$ ; where  $M_{22}$  is set of all  $2 \times 2$  Matrices.

Let  $M_{22}$  matrix in non se empty set of matrix.

and,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C = \begin{bmatrix} i & j \\ k & l \end{bmatrix} \in M_{22}$$

(1) Closure: Let  $A, B \in M_{22}$

$$\therefore A + B = \begin{bmatrix} a + e & b + f \\ c + g & d + h \end{bmatrix} \in M_{22}$$

(2) Associativity:

$$\rightarrow A * (B * C) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} ci + fk & ej + kl \\ gi + hk & gj + hl \end{bmatrix}$$

$$= \begin{bmatrix} aei + afk + bgi + bhk \\ cei + cfk + dgi + dhk \end{bmatrix}$$

$$\begin{bmatrix} aei + afe + bgi + bhe \\ cej + cfe + dgi + dhe \end{bmatrix} \quad \text{--- (1)}$$



$$\rightarrow (A * B) * C = \begin{bmatrix} ae+by & af+bh \\ ce+dy & cf+hd \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix}$$

$$= \begin{bmatrix} aeit + bgi + afk + bhk \\ cei + dgi + cfk + dhk \end{bmatrix}$$

$$\begin{bmatrix} aef + bgi + afe + bhe \\ cej + dgi + cfe + hde \end{bmatrix} \quad \text{--- (2)}$$

From, eq<sup>n</sup> 1 = 2

(3) Identity:

$$\text{Let } e = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} \in M_{22}$$

$$\therefore A * e = A$$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\therefore e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Identity element is  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(4) Inverse:

$$\text{Let } a * a^{-1} = e$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Hence,  $M_{22}$  is a Algebraic structure.

- 5 Prove that the set of cubic root of unity under multiplication is Abelian group. Find the order of the group and order of each element.

Here, we want to show that cubic roots of unity under multiplication, For that,

$$\therefore x^3 = 1$$

$$\therefore x^3 - 1 = 0$$

$$\therefore (x-1)(x^2+x+1) = 0$$

$$\therefore x = 1 \text{ or } x = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\therefore x = 1, w, w^2$$

$$\text{Hence, } G = \{1, w, w^2\}$$

→ Composition table:

*	1	w	w <sup>2</sup>
1	1	w	w <sup>2</sup>
w	w	w <sup>2</sup>	1
w <sup>2</sup>	w <sup>2</sup>	1	w

(1) Closure: From Composition table,  
For every  $a * b \in G$ ,  $\forall a, b \in G$

(2) Associativity: From Composition table, it is clear that  $G$  is associative.

(3) Identity: From Composition table, identity element is 1.

(4) Inverse: From Composition table,

Inverse of 1 is 1.

Inverse of w is w<sup>2</sup>

Inverse of w<sup>2</sup> is w.

Hence, Given set  $G$  is group under the multiplication.

6 Find the order of the group and order of each element for,

(a)  $(\mathbb{Z}_7, \times_7)$

$$\mathbb{Z}_7 = \{[1], [2], [3], [4], [5], [6]\}$$

Order of Group = 6

$$g = [1] = [1]^1 \Rightarrow o([1]) = 1$$

$$g = [2] = [2]^3 \Rightarrow o([2]) = 3$$

$$g = [3] = [3]^6 \Rightarrow o([3]) = 6$$

$$g = [4] = [4]^4 \Rightarrow o([4]) = 4$$

$$g = [5] = [5]^3 \Rightarrow o([5]) = 3$$

$$g = [6] = [6]^2 \Rightarrow o([6]) = 2$$

(b)  $(\mathbb{Z}_5, +_6)$

$$\mathbb{Z}_5 = \{[1], [2], [3], [4], [0]\}$$

Order of Group = 5

$$g = [1] = [1]^5 \Rightarrow o([1]) = 5$$

$$g = [2] = [2]^5 \Rightarrow O([2]) = 5$$

$$g = [3] = [3]^5 \Rightarrow O([3]) = 5$$

$$g = [4] = [4]^5 \Rightarrow O([4]) = 5$$

$$g = [0] = [0]^5 \Rightarrow O([0]) = 5$$

Brain Spot

## \* Task: 3: Sub Group and Coset of a Sub Group.

1 Define following Terms.

(i) Sub group

Let  $G$  is a Group under the binary operation and  $a \subset G$ . If  $a$  is called subgroup of  $G$ , then  $a$  Subgroup follow all the property of Group.

(ii) Improper Sub Group:

Let  $G$  is a Group under the binary operation and  $G$  has to Sub Group which is Group itself and identity, they Group are called Improper Sub Group.

(iii) Coset:

Let  $H$  be a Subgroup of  $G$  then the set  $\{a * h; h \in H\}$  is called the left coset generated by  $a$  and  $H$ . It is denoted by  $aH$ .

Left Coset =  $a + H$

Right Coset =  $H + a$

(iv) Index of Sub group:

Let  $G$  is a Group and  $H$  is a subgroup, then number of the coset of subgroup is known as Index of Subgroup.

2 State Lagrange's Theorem with the condition.

-> Lagrange's Theorem:

The order of each sub group a finite Group  $G$  is a divider of Order of  $G$ .

Or

Let  $G$  be a Group with  $O(G) = n$  and  $H$  be a subgroup with  $O(H) = n$  then  $O(H) \mid O(G)$ .

- Condition:

This are the two condition of Lagrange's Theorem.

1  $a, b \in H \Rightarrow a * b \in H$

2  $b \in H \Rightarrow b^{-1} \in H$

3 Find all Subgroups of the following groups:

(a)  $(Z_6, +_6)$

$Z_6 = \{[0], [1], [2], [3], [4], [5]\}$

- Composition table:

$+_6$	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[1]	[2]	[3]	[4]	[5]
[1]	[1]	[2]	[3]	[4]	[5]	[0]
[2]	[2]	[3]	[4]	[5]	[0]	[1]
[3]	[3]	[4]	[5]	[0]	[1]	[2]
[4]	[4]	[5]	[0]	[1]	[2]	[3]
[5]	[5]	[0]	[1]	[2]	[3]	[4]

From the table,

1 Closure:  $\forall a, b \in Z_6$   
 $a +_6 b \in Z_6$

2 From the table, We can see that  $(Z_6, +_6)$  is associative.



3 Identity : 0 is the identity element of  $(\mathbb{Z}_6, +_6)$ .

4 Existence of Inverse :

Inverse of 0 is 0.

Inverse of 1 is 5 and inverse of 5 is 1

Inverse of 2 is 4

Inverse of 3 is 3

Inverse of 4 is 2

$(\mathbb{Z}_6, +_6)$  follows all the properties of Group.

- Subgroup :

$(\mathbb{H}_3, +_6)$  is a Subgroup of  $\{0, 1, 3, 5\}$

$(\mathbb{H}_4, +_6)$  is a Subgroup of  $\{0, 2, 4\}$

$(\mathbb{H}_5, +_6)$  is a Subgroup of  $\{0, 5\}$

$$(b) G = (\{\pm 1, \pm i\}, \times)$$

$$G = \{1, -1, +i, -i\}$$

- Composition table:-

X	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

From the table,

ci) Closure:  $\forall a, b \in G, a \times b \in G$

cii) From the table, we can see that  $G$  is a Associative.

ciii) From the table, Identity element is 1.

civ) Inverse:

Inverse of 1 is 1

Inverse of -1 is -1

Inverse of -i is +i

Inverse of i is -i

- Sub Group:

$(H_3, \times)$  where  $H = \{1, -1\}$

~~$(H_4, \times)$  where  $H = \{1, i, -1, -i\}$~~

4 Prove that  $H$  is sub group of a give Group  $G$  in each of the following. Find all coset of  $H$ . also Find index of  $H (G:H)$ .

(i)  $G = (Z_6, +_6)$  and  $H = \{0, 3\}$

$\Rightarrow$  For Group  $G = (Z_6, +_6)$

Composition table:

$$Z_6 = \{[0], [1], [2], [3], [4], [5]\}$$

$+_6$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$	$[5]$
$[0]$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$	$[5]$
$[1]$	$[1]$	$[2]$	$[3]$	$[4]$	$[5]$	$[0]$
$[2]$	$[2]$	$[3]$	$[4]$	$[5]$	$[0]$	$[1]$
$[3]$	$[3]$	$[4]$	$[5]$	$[0]$	$[1]$	$[2]$
$[4]$	$[4]$	$[5]$	$[0]$	$[1]$	$[2]$	$[3]$
$[5]$	$[5]$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$

From the table,

(i) Closure:  $\forall a, b \in G, a +_6 b \in G$

(ii) From the table, we can see that  $(Z_6, +_6)$  is associative.

ciii) Identity element is  $[0]$ .

civ) Inverse:

Inverse of 0 is 0

Inverse of 1 is 5

Inverse of 2 is 4

Inverse of 3 is 3

Inverse of 4 is 2

Inverse of 5 is 1

Here,  $(\mathbb{Z}_6, +_6)$  is a Group.

$\Rightarrow$  For  $H = \{0, 3\}$

$+_6$	0	3
0	0	3
3	3	0

From the table,

ci) Closure:  $\forall a, b \in H, a +_6 b \in H$

cii) From the table, we can see that,  $(H, +_6)$  is associative

ciii) Identity Element is  $[0]$

civ) Inverse of 0 is 0

Inverse of 3 is 3

So,  $(H, +_6)$  is a Sub Group of  $(Z_6, +_6)$ .

$\Rightarrow$  For Coset

- For  $0 \in Z_6$

$$\begin{aligned} \text{Left coset} &= \{0+_6 0, 0+_6 3\} \\ &= \{0, 3\} \end{aligned}$$

$$\begin{aligned} \text{Right coset} &= \{0+_6 0, 3+_6 0\} \\ &= \{0, 3\} \end{aligned}$$

- For  $1 \in Z_6$

$$\begin{aligned} \text{Left coset} &= \{1+_6 0, 1+_6 3\} \\ &= \{1, 4\} \end{aligned}$$

$$\begin{aligned} \text{Right coset} &= \{0+_6 1, 3+_6 1\} \\ &= \{1, 4\} \end{aligned}$$

- For  $2 \in Z_6$

$$\begin{aligned} \text{Left coset} &= \{2+_6 0, 2+_6 3\} \\ &= \{2, 5\} \end{aligned}$$

$$\begin{aligned} \text{Right coset} &= \{0+_6 2, 3+_6 2\} \\ &= \{2, 5\} \end{aligned}$$

- For  $3 \in Z_6$

$$\text{Left coset} = \{3 +_6 0, 3 +_6 3\} \\ = \{3, 0\}$$

$$\text{Right coset} = \{0 +_6 3, 3 +_6 3\} \\ = \{3, 0\}$$

- For  $4 \in Z_6$

$$\text{Left coset} = \{4 +_6 0, 4 +_6 3\} \\ = \{4, 1\}$$

$$\text{Right coset} = \{0 +_6 4, 3 +_6 4\} \\ = \{4, 1\}$$

- For  $5 \in Z_6$

$$\text{Left coset} = \{5 +_6 0, 5 +_6 3\} \\ = \{5, 2\}$$

$$\text{Right coset} = \{0 +_6 5, 3 +_6 5\} \\ = \{5, 2\}$$

Here, Distinct cosets = 3

Index of  $H(G; H) = 3$

cii)  $G = (\{ \pm 1, \pm i \}, \times)$ ,  $H = (\{ 1, -1 \}, \times)$

$\Rightarrow$  For  $G = \{ 1, -1, i, -i \}$

Composition table:

X	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

From the table,

(i) Closure:  $\forall a, b \in G, a \times b \in G$

(ii) From the table we can see that  $G$  is a Associative.

(iii) Identity element is 1.

(iv) Inverse:

Inverse of 1 is 1

Inverse of -1 is -1

Inverse of i is -i

Inverse of -i is i

Here, All the Property is follow.  
So,  $G$  is a Group.

$\Rightarrow$  For  $H = \{1, -1, x\}$

$x$	$1$	$-1$
$1$	$1$	$-1$
$-1$	$-1$	$1$

From the table,

(i) Closure:  $\forall a, b \in H, a \times b \in H$

(ii) From the table, we can see that  $H$  is associative.

(iii) Identity element is  $1$ .

(iv) Inverse:

Inverse of  $1$  is  $1$

Inverse of  $-1$  is  $-1$

Hence,  $H$  is a Subgroup of  $G$ .

$\Rightarrow$  For Coset:

- For  $1 \in G$

$$\begin{aligned} \text{Left Coset} &= \{1 \times 1, 1 \times -1\} \\ &= \{1, -1\} \end{aligned}$$



$$\text{Right Coset} = \{1 \times 1, -1 \times 1\} \\ = \{1, -1\}$$

- For  $-1 \in G$

$$\text{Left Coset} = \{-1 \times 1, -1 \times -1\} \\ = \{-1, 1\}$$

$$\text{Right Coset} = \{1 \times -1, -1 \times -1\} \\ = \{-1, 1\}$$

- For  $i \in G$

$$\text{Left Coset} = \{i \times 1, i \times -1\} \\ = \{i, -i\}$$

$$\text{Right Coset} = \{1 \times i, -1 \times i\} \\ = \{i, -i\}$$

- For  $-i \in G$

$$\text{Left Coset} = \{-i \times 1, -i \times -1\} \\ = \{-i, i\}$$

$$\text{Right Coset} = \{1 \times -i, -1 \times -i\} \\ = \{-i, i\}$$

Here, Distinct Coset = 2

$$\text{Index of } H(G:H) = 2$$

(iii)  $G = (\mathbb{Z}, +)$  and  $H = (2\mathbb{Z}, +)$

$\Rightarrow$  For Group  $G$

$$G = \{ \dots, -1, 0, 1, 2, \dots \}$$

(i) Closure:  $\forall a, b \in G, a + b \in G$ ;

(ii) Associative:  $\forall a, b, c \in G$

$$\therefore (a + b) + c = a + (b + c) \in G$$

$\therefore$  It is Associative.

(iii) Identity element is 0.

(iv) Inverse:  $\forall a \in G$

$$a + b = e$$

$$\therefore a + b = 0$$

$$\therefore b = -a$$

So, All Property is follow by the  $G$  So,  $G$  is a Group.

$\Rightarrow$  For Subgroup  $H$

$$H = \{ \dots, -2, 0, 2, 4, 6, \dots \}$$

(i) Closure:  $\forall a, b \in H, a + b \in H$

(ii) Associative:  $\forall a, b, c \in H$

$$\therefore (a + b) + c = a + (b + c)$$

$\therefore$  It is associative

(iii) Identity element is 0

(iv) Inverse:  $\forall a \in H$

$$a + b = e$$

$$\therefore a + b = 0$$

$$\therefore a = -b$$

So, Here,  $H$  is a Subgroup of  $G$ .

$\Rightarrow$  Coset:

- For  $0 \in \mathbb{Z}$

Left Coset -  $H$

$$0 + H = \{ \dots, -4, -2, 0, 2, 4, \dots \}$$

Right Coset

$$H + 0 = \{ \dots, -4, -2, 0, 2, 4, \dots \}$$

- For  $1 \in \mathbb{Z}$

$$L.C = 1 + H = \{ \dots, -3, -1, 1, 3, \dots \}$$

$$R.C = H + 1 = \{ \dots, -3, -1, 1, 3, \dots \}$$

- For  $2 \in \mathbb{Z}$

$$L.C = 2 + H = \{ \dots, -2, 0, 2, \dots \}$$

$$R.C = H + 2 = \{ \dots, -2, 0, 2, \dots \}$$

- For  $3 \in \mathbb{Z}$

$$L.C = 3 + H = \{ \dots, -3, -1, 1, 3, \dots \} = H + 1$$

$$R.C = H + 3 = \{ \dots, -3, -1, 1, 3, \dots \} = H + 1$$

- For  $4 \in \mathbb{Z}$

$$L.C = 4 + H = \{ \dots, -2, 0, 2, \dots \} = H + 2$$

$$R.C = H + 4 = \{ \dots, -2, 0, 2, \dots \} = H + 2$$

- Distinct Coset =  $\{ H, 1 + H, 2 + H \}$

- Index of  $H(G:H) = 3$

## \* Task - 4: Various Group - 1

1 Define the following terms.

(i) Permutation Group

A Permutation Group is a finite group  $G$  whose elements are permutation of a given set and whose group operation is composition of Permutation of  $G$ .

(ii) Cyclic Group:

Let  $G$  be a Group and  $a \in G$  then, for some  $a \in G$  every element of  $G$  is of the form  $a^n$  where  $n$  is some integer that is,

$$G = \{a^n, n \in \mathbb{Z}\}$$

(iii) Generator:

The element that generates a cyclic group, these elements are called generator.

2 State Cayley's Theorem. Prove that  $(S_3, \circ)$  is non-abelian permutation Group.

Cayley's Theorem: IF  $G$  is a Group then there exists a subgroup  $H$  of  $Sym(G)$  such that  $G$  is isomorphic to  $H$ .

$\rightarrow S_3 = \{P_1, P_2, P_3, P_4, P_5, P_6\}$

Composition table,

	$P_1$	$P_2$	$P_3$	$P_4$	$P_5$	$P_6$
$P_1$	$P_1$	$P_2$	$P_3$	$P_4$	$P_5$	$P_6$
$P_2$	$P_2$	$P_5$	$P_6$	$P_5$	$P_4$	$P_3$
$P_3$	$P_3$	$P_5$	$P_1$	$P_6$	$P_2$	$P_4$
$P_4$	$P_4$	$P_6$	$P_5$	$P_1$	$P_3$	$P_2$
$P_5$	$P_5$	$P_3$	$P_4$	$P_2$	$P_6$	$P_1$
$P_6$	$P_6$	$P_4$	$P_2$	$P_3$	$P_1$	$P_5$

From the table,

- (i) Closure:  $\forall P_i, P_j \in S_3 \rightarrow P_i \cdot P_j \in S_3$
- (ii) Associative: From the table, it is clear that  $S_3$  is associative.
- (iii) Identity:  $P_1$  is the identity element.

civ) Inverse: From table inverse of  $P_1, P_2, P_3, P_4, P_5, P_6$  are  $P_1, P_2, P_3, P_4, P_5, P_6$  respectively.

cv) Let  $P_3, P_4 \in S_3$

$$\text{So, } P_3 \cdot P_4 = P_5 \text{ and } P_4 \cdot P_3 = P_6$$

$$\therefore P_3 \cdot P_4 \neq P_4 \cdot P_3$$

Hence,  $S_3$  is not a abelian Group.

3 Prove that following structure is cyclic. Find all its generators.

$$ci) G = (\{1, -1, i, -i\}, \times)$$

Here,  $a = -1$  and  $a = 1$  is not contain the all Group element such that  $a = -1$  and  $a = 1$  is not generators.

$$a = i \Rightarrow (i)^1 = i$$

$$(i)^2 = -1$$

$$(i)^3 = -i$$

$$(i)^4 = 1$$

$$a = -i \Rightarrow (-i)^1 = -i$$

$$(-i)^2 = -1$$

$$(-i)^3 = i$$

$$(-i)^4 = 1$$

$a = -i$  and  $a = i$  contain all Group element by taking different power.

Such that  $a = i$  and  $-i$  is Generator.

$\therefore \langle G, X \rangle$  is a Cyclic Group generated by  $i$  and  $-i$

(ii)  $(G = \mathbb{Z}_8, +_8)$

$$G = \{[0], [1], [2], [3], [4], [5], [6], [7]\}$$

- For  $[0] \in G$

$$\therefore 0^1 = 0$$

$$\therefore 0^2 = 0$$

$[0]$  is not Generator.

- For  $[1] \in G$

$$\therefore [1]^1 = [1]$$

$$\therefore [1]^2 = [2]$$

$$\therefore [1]^3 = [3]$$

$$\therefore [1]^4 = [4]$$

$$\therefore [1]^5 = [5]$$

$$\therefore [1]^6 = [6]$$

$$\therefore [1]^7 = [7]$$

$$\therefore [1]^0 = [0]$$



$[1]$  is Generator

- For  $[2] \in G$

$$\therefore [2]^1 = [2]$$

$$\therefore [2]^2 = [4]$$

$$\therefore [2]^3 = [6]$$

$$\therefore [2]^0 = [0]$$

$\therefore [2]$  is not Generator.

- For  $[3] \in G$ ,

$$\therefore [3]^1 = [3]$$

$$\therefore [3]^2 = [6]$$

$$\therefore [3]^3 = [1]$$

$$\therefore [3]^4 = [4]$$

$$\therefore [3]^5 = [7]$$

$$\therefore [3]^6 = [2]$$

$$\therefore [3]^7 = [5]$$

$$\therefore [3]^8 = 0$$

$\therefore [3]$  is Generator

- For  $[4] \in G$ ,

$$\therefore [4]^1 = 4$$

$$\therefore [4]^2 = 0$$

$$\therefore [4]^3 = 4$$

$\therefore [4]$  is not Generator

- For  $[5] \in G$

$$\therefore [5]^1 = [5]$$

$$\therefore [5]^2 = [2]$$

$$\therefore [5]^3 = [7]$$

$$\therefore [5]^4 = [4]$$

$$\therefore [5]^5 = [1]$$

$$\therefore [5]^6 = [6]$$

$$\therefore [5]^7 = [3]$$

$$\therefore [5]^8 = [0]$$

$\therefore [5]$  is a Generator.

- For  $[6] \in G$

$$\therefore [6]^1 = [6]$$

$$\therefore [6]^2 = [4]$$

$$\therefore [6]^3 = [2]$$

$$\therefore [6]^4 = [0]$$

$\therefore [6]$  is not Generator

- For  $[7] \in G$

$$\therefore [7]^1 = [7]$$

$$\therefore [7]^2 = [6]$$

$$\therefore [7]^3 = [5]$$

$$\therefore [7]^4 = [4]$$

$$\therefore [7]^5 = [3]$$

$$\therefore [7]^6 = [2]$$

$$\therefore [7]^7 = [1]$$

$$\therefore [7]^0 = 0$$

$\therefore [7]$  is not Generator.

$\therefore \langle \mathbb{Z}_7, + \rangle$  is a Cyclic Group  
generator by 1, 5, 7, 3.

4 Find all generators of a cyclic group of order 16.

By Lagrange's Theorem

$$H = \{1, 2, 4, 8, 16\}$$

$$\rightarrow \text{IF } o(CH) = 1 \Rightarrow H = \langle a^{16/1} \rangle$$

$$H_1 = \{a^{16}\}$$

$$\rightarrow \text{IF } o(CH) = 2 \Rightarrow H = \langle a^{16/2} \rangle$$

$$H_2 = \{a^{16}, a^8\}$$

$$\rightarrow \text{IF } o(CH) = 4 \Rightarrow H = \langle a^{16/4} \rangle$$

$$H_3 = \{a^{16}, a^8, a^{12}, a^4\}$$

$$\rightarrow \text{IF } o(CH) = 8 \Rightarrow H = \langle a^{16/8} \rangle$$

$$H_4 = \{a^2, a^4, a^6, a^8, a^{10}, a^{12}, a^{14}, a^{16}\}$$

$$\rightarrow \text{IF } o(CH) = 16 \Rightarrow H = \langle a^{16/16} \rangle$$

Page No. \_\_\_\_\_  
Date: / /

$$H_4 = \{ a^1, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}, a^{15}, a^{16} \}$$

→ Now we Find Order of each element,

$$a^1 = (a^1)^{16} = e \Rightarrow o(a^1) = 16$$

$$a^2 = (a^2)^8 = e \Rightarrow o(a^2) = 8$$

$$a^3 = (a^3)^{16} = e \Rightarrow o(a^3) = 16$$

$$a^4 = (a^4)^4 = e \Rightarrow o(a^4) = 4$$

$$a^5 = (a^5)^{16} = e \Rightarrow o(a^5) = 16$$

$$a^6 = (a^6)^8 = e \Rightarrow o(a^6) = 8$$

$$a^7 = (a^7)^{16} = e \Rightarrow o(a^7) = 16$$

$$a^8 = (a^8)^2 = e \Rightarrow o(a^8) = 2$$

$$a^9 = (a^9)^{16} = e \Rightarrow o(a^9) = 16$$

$$a^{10} = (a^{10})^8 = e \Rightarrow o(a^{10}) = 8$$

$$a^{11} = (a^{11})^{16} = e \Rightarrow o(a^{11}) = 16$$

$$a^{12} = (a^{12})^4 = e \Rightarrow o(a^{12}) = 4$$

$$a^{14} = (a^{14})^8 = e \Rightarrow o(a^{14}) = 8$$

$$a^{13} = (a^{13})^{16} = e \Rightarrow o(a^{13}) = 16$$

$$a^{15} = (a^{15})^{16} = e \Rightarrow o(a^{15}) = 16$$

$$a^{16} = (a^{16})^1 = e \Rightarrow o(a^{16}) = 1$$

Here,  $a^1, a^5, a^7, a^{11}, a^{13}$   
are Generator of Group  $G$ .

5 IF  $\sigma, \alpha$  and  $\tau \in S_6$  defined by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 6 & 5 & 4 \end{pmatrix}$$

$$\text{and } \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 2 & 5 & 1 \end{pmatrix}.$$

Find the following.

(1)  $\sigma^2 \tau$

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 4 & 5 & 6 \end{pmatrix}$$

$$\sigma^2 \cdot \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 2 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 2 & 5 & 1 \end{pmatrix}$$

$$(2) \tau^{-1} \tau$$

$$\therefore \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 6 & 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & b & c & d & e & f \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ c & d & b & f & e & a \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$\therefore \tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 1 & 2 & 5 & 4 \end{pmatrix}$$

$$\therefore \tau^{-1} \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 1 & 2 & 5 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 2 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 4 & 5 & 2 \end{pmatrix}$$

$$(3) O(\sigma)$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix}$$

Cycle form  $\rightarrow (1 \ 2 \ 3) (4) (5 \ 6)$

For order:  $\text{LCM}(3, 1, 2)$

$$O(\sigma) = 6$$

(4)  $O(\tau)$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 2 & 5 & 1 \end{pmatrix}$$

Cycle =  $(1\ 6)(2\ 4)(5)(3)$   
 For Order =  $LCM(2\ 2\ 1\ 1)$   
 $O(\tau) = 2$

$\Rightarrow$  Verify:

$$1\ (\sigma\eta)^{-1} = \eta^{-1}\sigma^{-1}$$

L.H.S. =  $\sigma\eta$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 6 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 3 & 6 & 1 & 5 \end{pmatrix}$$

$\therefore (\sigma\eta)^{-1} =$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 3 & 6 & 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & b & c & d & e & f \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$\therefore (\sigma \eta)^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 3 & 1 & 6 & 4 \end{pmatrix} \quad \text{--- (1)}$$

RHS.:

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & b & c & d & e & f \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 3 & 4 & 6 & 5 \\ & 1 & 2 & & & \end{pmatrix}$$

$$\sigma^{-1} \eta^{-1}$$

$$\eta^{-1} \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 3 & 1 & 6 & 4 \end{pmatrix} \quad \text{--- (2)}$$

$\therefore$  LHS = RHS.

Hence,  $(\sigma \eta)^{-1} = \eta^{-1} \sigma^{-1}$

$$2 \quad \sigma \cdot (\eta \cdot \tau) = (\sigma \cdot \eta) \cdot \tau$$

LHS.

$$\eta \circ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 6 & 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 2 & 5 & 1 \end{pmatrix}$$



$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 4 & 5 & 6 \end{pmatrix}$$

$$\sigma \circ (\sigma \circ \tau) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix} \circ$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 4 & 5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 6 & 1 \end{pmatrix}$$

R.H.S. :

$$\sigma \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \sigma(\sigma(1)) & \sigma(\sigma(2)) & \sigma(\sigma(3)) & \sigma(\sigma(4)) \\ 5 & 6 \\ \sigma(\sigma(5)) & \sigma(\sigma(6)) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 5 & 6 & 2 \end{pmatrix} = \sigma$$

$$(\sigma \circ \sigma) \circ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \sigma(\tau(1)) & \sigma(\tau(2)) & \sigma(\tau(3)) & \sigma(\tau(4)) \\ 5 & 6 \\ \sigma(\tau(5)) & \sigma(\tau(6)) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 6 & 1 \end{pmatrix}$$

Here, LHS = RHS.

Hence,  $\sigma \circ (\psi \circ \tau) = (\sigma \circ \psi) \circ \tau$   
is prove.

Page No. \_\_\_\_\_  
Date - / /

## \* Task : 5: Various Groups - II

1 Define the following term

(i) Normal Sub Group:

A subgroup  $H$  of a group  $G$  is said to be a normal subgroup of  $G$  if

$$\text{Left Coset} = \text{Right Coset}$$

$$\therefore aH = Ha$$

Every abelian group is a normal subgroup.

(ii) Quotient Group:

If  $H$  is a normal subgroup of  $G$  then the set of all left coset of  $G$  forms a group with respect to multiplication left coset and defined as,

$$(aH)bH = abH$$

② Prove that Sub group  $H = \{P_1, P_2, P_3\}$  is normal for the group  $(S_3, \circ)$

For Finding the Subgroup.

$e$	$P_1$	$P_2$	$P_3$	$P_4$	$P_5$	$P_6$
$P_1$	$P_1$	$P_2$	$P_3$	$P_4$	$P_5$	$P_6$
$P_2$	$P_2$	$P_1$	$P_6$	$P_5$	$P_4$	$P_3$
$P_3$	$P_3$	$P_5$	$P_1$	$P_6$	$P_2$	$P_4$
$P_4$	$P_4$	$P_6$	$P_5$	$P_1$	$P_3$	$P_2$
$P_5$	$P_5$	$P_3$	$P_4$	$P_2$	$P_6$	$P_1$
$P_6$	$P_6$	$P_4$	$P_2$	$P_3$	$P_1$	$P_5$

→ For normal Group, Always,  
Right Coset = Left Coset

- Left Coset

- For  $P_1 \in H$

$$\begin{aligned} \text{Right Coset} &= (P_1, P_2, P_3) \cdot P_1 \\ &= (P_1 \cdot P_1, P_2 \cdot P_1, P_3 \cdot P_1) \\ &= (P_1, P_2, P_3) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Left Coset} &= P_1 \cdot (P_1, P_2, P_3) \\ &= (P_1 \cdot P_1, P_1 \cdot P_2, P_1 \cdot P_3) \\ &= (P_1, P_2, P_3) - \textcircled{2} \end{aligned}$$

$$e q^n 1 = 2$$

→ For  $P_2 \in G$

$$\begin{aligned} \text{Right Coset} &= (P_1, P_2, P_3) \cdot P_2 \\ &= (P_2, P_1, P_6) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Left Coset} &= P_2 \cdot (P_1, P_2, P_3) \\ &= (P_2, P_1, P_6) - \textcircled{2} \end{aligned}$$

$$eq^n \quad 1 = 2$$

→ For  $P_3 \in G$

$$\begin{aligned} \text{Right Coset} &= (P_1, P_2, P_3) \cdot P_3 \\ &= (P_3, P_6, P_1) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Left Coset} &= P_3 \cdot (P_1, P_2, P_3) \\ &= (P_3, P_6, P_1) - \textcircled{2} \end{aligned}$$

$$eq^n \quad 1 = 2$$

- For  $P_4 \in G$

$$\begin{aligned} \text{Right Coset} &= (P_1, P_2, P_3) \cdot P_4 \\ &= (P_4, P_6, P_5) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Left Coset} &= P_4 \cdot (P_1, P_2, P_3) \\ &= (P_4, P_6, P_5) - \textcircled{2} \end{aligned}$$

$$eq^n \quad 1 = 2$$

- For  $P_5 \in G$

$$\begin{aligned} \text{Right coset} &= (P_1, P_2, P_3) \cdot P_5 \\ &= (P_5, P_3, P_4) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Left coset} &= P_5 \cdot (P_1, P_2, P_3) \\ &= (P_5, P_3, P_4) - \textcircled{2} \end{aligned}$$

- For  $P_6 \in G$

$$\begin{aligned} \text{Right coset} &= (P_1, P_2, P_3) \cdot P_6 \\ &= (P_6, P_4, P_2) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Left coset} &= P_6 \cdot (P_1, P_2, P_3) \\ &= (P_6, P_4, P_2) - \textcircled{2} \end{aligned}$$

Here,  $H$  is be a normal  
Sub group of  $G$

$$G/H = \{H_1, H_2, H_3, H_4, H_5, H_6\}$$

$$G/H = \{HP_1, HP_2, HP_3, HP_4, HP_5, HP_6\}$$

- Check whether Sub Group  $H = \{P_1, P_4\}$  is normal for the Group  $(S_3, \circ)$ ?

For normal Group Always,

We have to check,

Left Coset = Right Coset.

- For  $P_1 \in G$

$$\begin{aligned} \text{Right Coset} &= (P_1, P_2, P_3) \cdot P_1 \\ &= (P_1, P_2, P_3) \quad \text{--- (1)} \end{aligned}$$

~~Left coset =  $P_1 \cdot CP_1$~~

Right coset =  $(P_1, P_4) \cdot P_1$   
 $= (P_1, P_4) - \textcircled{1}$

Left coset =  $P_1 \cdot (P_1, P_4)$   
 $= (P_1, P_4) - \textcircled{2}$

$eq^n 1 = 2$

- For  $P_2 \in G$

Right coset =  $(P_1, P_4) \cdot P_2$   
 $= (P_2, P_5) - \textcircled{1}$

Left coset =  $P_2 \cdot (P_1, P_4)$   
 $= (P_2, P_6) - \textcircled{2}$

$eq^n 1 \neq 2$

Here, For  $P_2$ ,  $H = \{P_1, P_4\}$   
 is not be a normal subgroup

Thus,  $H = \{P_1, P_4\}$  is not be a  
 normal subgroup of  $(S_3, \circ)$



3 Prove that Sub Group  $H = \{1, -1\}$  is normal for the Group  $G = \{1, -1, i, -i\}$

→ Composition table

x	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	1	-1
-i	-i	i	-1	1

→ For normal Group,

$$\text{Left Coset} = \text{Right Coset}$$

- For  $1 \in G$

$$\begin{aligned} \text{Left Coset} &= 1 \cdot (1, -1) \\ &= (1, -1) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Right Coset} &= (1, -1) \cdot 1 \\ &= (1, -1) - \textcircled{2} \end{aligned}$$

$$\text{eq}^n \neq 2$$

- For  $-1 \in G$

$$\begin{aligned} \text{Left Coset} &= -1 \cdot (1, -1) \\ &= (-1, 1) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Right Coset} &= (1, -1) \cdot -1 \\ &= (-1, 1) - \textcircled{2} \end{aligned}$$

$$\text{eq}^n \quad 1 = 2$$

- For  $i \in G$

$$\begin{aligned} \text{Left Coset} &= i \cdot (1, -1) \\ &= (i, -i) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Right Coset} &= (1, -1) \cdot i \\ &= (i, -i) - \textcircled{2} \end{aligned}$$

$$\text{eq}^n \quad 1 = 2$$

- For  $-i \in G$

$$\begin{aligned} \text{Left Coset} &= -i(1, -1) \\ &= (-i, i) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Right Coset} &= (1, -1) \cdot -i \\ &= (-i, i) - \textcircled{2} \end{aligned}$$

$$\text{eq}^n \quad 1 = 2$$

Here, For all the Group  $G$   
Left Coset = Right Coset

Thus,  $H = \{1, -1\}$  is a normal subgroup For  $S_3$   $G$ .

4 Prove that  $G/H$  is Quotient Group For the following Group  $G$  and normal subgroup  $H$ , Using composition table,

(i)  $G = (\{\pm 1, \pm i\}, \times)$   $H = (\{1, -1\}, \times)$

-> Composition table For  $G$

$$G = \{1, -1, i, -i\}$$

$\times$	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

Composition table For  $H$

$$H = \{1, -1\}$$

$\times$	1	-1
1	1	-1
-1	-1	1

-> For Normal Subgroup,

$$\text{Left Coset} = \text{Right Coset}$$

-> For  $1 \in G$

$$\begin{aligned} \text{Left coset} &= 1 \cdot (1, -1) \\ &= (1, -1) \\ &= (1, -1) \cdot 1 \\ &= \text{Right coset} \end{aligned}$$

-> For  $-1 \in G$

$$\begin{aligned} \text{Left Coset} &= -1 \cdot (1, -1) \\ &= (-1, 1) \\ &= (1, -1) \cdot -1 \\ &= \text{Right Coset} \end{aligned}$$

-> For  $i \in G$

$$\begin{aligned} \text{Left coset} &= i \cdot (1, -1) \\ &= (i, -i) \\ &= (1, -1) \cdot i \\ &= \text{Right coset} \end{aligned}$$

-> For  $-i \in G$

$$\begin{aligned} \text{Left Coset} &= -i \cdot (1, -1) \\ &= (-i, i) \\ &= (1, -1) \cdot -i \\ &= \text{Right Coset} \end{aligned}$$

$$\therefore G/H = \{H, H(i), H(-1), H(-i)\}$$

X	H	-H	Hi	H(-i)
H	H	-H	Hi	H-i
-H	-H	H	-Hi	Hi
Hi	iH	-Hi	H	H
H(-i)	-iH	-Hi	H	H

From Composition table,

1 Closure:

$$\forall a, b \in G, a \times b \in G, H \times H \in G$$

2 Associative:  $\forall a, b, c \in G$

$$(a \times b) \times c = a \times (b \times c)$$

$$\therefore (H \times H) \times (-H) = H \times (H \times -H)$$

3 Existence of Identity:

H is an Identity element,

4 Inverse: All elements are exists Inverse.

Therefore, all Property is follow then  $\in$  it is a Group.

(ii)  $G = (S_3, 0)$  and  $H = \{P_1, P_2, P_3\}$

→ Composition table For  $G$

0	$P_1$	$P_2$	$P_3$	$P_4$	$P_5$	$P_6$
$P_1$	$P_1$	$P_2$	$P_3$	$P_4$	$P_5$	$P_6$
$P_2$	$P_2$	$P_1$	$P_6$	$P_5$	$P_4$	$P_3$
$P_3$	$P_3$	$P_5$	$P_1$	$P_6$	$P_2$	$P_4$
$P_4$	$P_4$	$P_6$	$P_5$	$P_1$	$P_3$	$P_2$
$P_5$	$P_5$	$P_3$	$P_4$	$P_2$	$P_6$	$P_1$
$P_6$	$P_6$	$P_4$	$P_2$	$P_3$	$P_1$	$P_5$

For Normal Sub Group

Left coset = Right coset.

- For  $P_1, G/H$

$$\begin{aligned} \text{Right coset} &= (P_1, P_2, P_3) \cdot P_1 \\ &= (P_1, P_2, P_3) - (1) \end{aligned}$$

$$\begin{aligned} \text{Left coset} &= P_1 \cdot (P_1, P_2, P_3) \\ &= (P_1 \cdot P_1, P_1 \cdot P_2, P_1 \cdot P_3) \\ &= (P_1, P_2, P_3) - (2) \end{aligned}$$

From eq<sup>n</sup> 1 = 2

→ For  $P_2 \in G$

$$\begin{aligned} \text{Right Coset} &= (P_1, P_2, P_3) \cdot P_2 \\ &= (P_2, P_1, P_6) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Left Coset} &= P_2 \cdot (P_1, P_2, P_3) \\ &= (P_2, P_1, P_6) - \textcircled{2} \end{aligned}$$

$$eq^n \quad 1 = 2$$

→ For  $P_3 \in G$

$$\begin{aligned} \text{Right Coset} &= (P_1, P_2, P_3) \cdot P_3 \\ &= (P_3, P_6, P_1) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Left Coset} &= P_3 \cdot (P_1, P_2, P_3) \\ &= (P_3, P_6, P_1) - \textcircled{2} \end{aligned}$$

$$eq^n \quad 1 = 2$$

- For  $P_4 \in G$

$$\begin{aligned} \text{Right Coset} &= (P_1, P_2, P_3) \cdot P_4 \\ &= (P_4, P_6, P_5) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Left Coset} &= P_4 \cdot (P_1, P_2, P_3) \\ &= (P_4, P_6, P_5) - \textcircled{2} \end{aligned}$$

$$eq^n \quad 1 = 2$$

- For  $P_5 \in G$

$$\begin{aligned} \text{Right coset} &= (P_1, P_2, P_3) \cdot P_5 \\ &= (P_5, P_3, P_4) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Left coset} &= P_5 \cdot (P_1, P_2, P_3) \\ &= (P_5, P_3, P_4) - \textcircled{2} \end{aligned}$$

- For  $P_6 \in G$

$$\begin{aligned} \text{Right coset} &= (P_1, P_2, P_3) \cdot P_6 \\ &= (P_6, P_4, P_2) - \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Left coset} &= P_6 \cdot (P_1, P_2, P_3) \\ &= (P_6, P_4, P_2) - \textcircled{2} \end{aligned}$$

Here,  $H$  is be a normal sub group of  $G$

~~$$G/H = \{H1, H2, H3, H4, H5, H6\}$$~~

$$G/H = \{H.P_1, HP_2, HP_3, HP_4, HP_5, HP_6\}$$



-> Composition table:

$\circ$	$HP_1$	$HP_2$	$HP_3$	$HP_4$	$HP_5$	$HP_6$
$HP_1$	$HP_1$	$HP_2$	$HP_3$	$HP_4$	$HP_5$	$HP_6$
$HP_2$	$HP_2$	$HP_3$	$HP_6$	$HP_5$	$HP_4$	$HP_3$
$HP_3$	$HP_3$	$HP_5$	$HP_1$	$HP_6$	$HP_2$	$HP_4$
$HP_4$	$HP_4$	$HP_6$	$HP_5$	$HP_1$	$HP_3$	$HP_2$
$HP_5$	$HP_5$	$HP_3$	$HP_4$	$HP_2$	$HP_6$	$HP_1$
$HP_6$	$HP_6$	$HP_2$	$HP_2$	$HP_3$	$HP_1$	$HP_5$

From table,

(i) Closure: From table  $HP_i \cdot HP_j \in S_3$

(ii) From table, it is clear that  $S_3$  is associative.

(iii) Identity Element is  $HP_1$ .

(iv) Inverse: All the elements inverse are exists.

Thus,  $G/H$  is a Group.

\* Task: 6 Homomorphism of Groups

1 Define the following term,

(a) Homomorphism of Groups:

Let  $(G, *)$  and  $(G', *')$  be two groups then A mapping  $F: G \rightarrow G'$  is said to be a Homomorphism.

$$\text{if } F(a * b) = F(a) *' F(b)$$

(b) Kernel of Homomorphism:

Let  $(G, *)$  and  $(G', *')$  be two groups, A mapping  $F: G \rightarrow G'$  is a homomorphism then,

$$\text{Ker } F = \{a \in G, F(a) = e'\}$$

where  $e'$  = Identity element of  $G'$  Group.

2 Prove each of the following Function is homomorphism.

Find Kernel of the Homomorphism

(a)  $F: (\mathbb{C}, +) \rightarrow (\mathbb{R}, +)$  Defined by  
 $F(x + iy) = x$

Let  $a, b \in \mathbb{C}$ ,  $a, b \in \mathbb{R}$

$$\therefore F(a + ib) = a$$

$$\therefore F(a + ib) = F(a + ib)$$

Here  $F$  is a Homomorphism.

$\rightarrow$  Kernel of  $\otimes$  Homomorphism.

Here, Identity element of Group  $\mathbb{R}$  is  $0$  because given operation is addition.

$$\forall, F(x + iy) = x$$

$$\therefore F(0 + iy) = 0$$

$$\text{Ker } F = \{0 + iy\}$$

(b)  $F: (\mathbb{Z}_5, - \{0\}, X_5) \rightarrow (\mathbb{Z}_4, +_4)$

Defined by  $F = \{(1,0), (2,3), (3,1), (4,2)\}$

$$\text{Let } Z_5 = \{[1], [2], [3], [4]\}$$

$$Z_4 = \{[0], [1], [2], [3]\}$$

$$\text{Here, } F(1) = 0, \quad F(3) = 1 \\ F(2) = 3, \quad F(4) = 2$$

→ For Homomorphism,

- We take,  $[1] \in Z_5$   
 $[1] \in Z_4$

$$F(1 \times_5 1) = F(1) \\ = 0 \quad - \textcircled{1}$$

$$F(1) +_4 F(1) = 0 +_4 0 \\ = 0 \quad - \textcircled{2}$$

$$\text{From eq}^n, \quad 1 = 2$$

$$\therefore F(1 \times_5 1) = F(1) +_4 F(1)$$

- We take,  $[2] \in Z_5$   
 $[3] \in Z_4$

$$F(2 \times_5 3) = F(1) \\ = 0 \quad - \textcircled{1}$$

$$F(2) +_5 F(3) = F(1) \\ = 0 \quad - \textcircled{1}$$

From eq<sup>n</sup> 1 = 2

$$f(2 \times_5 3) = f(2) *_4 f(3)$$

Thus, the following Function  
is Homomorphism

-> Kernel of Homomorphism.

Here, Identity element for  $Z_4$   
is 0 because, Here operation  
is addition.

$$e' = 0$$

Ker  $F$ , We have to check

$$f(a) = e'$$

$$\therefore f(1) = 0$$

$$\therefore \text{Ker}(f) = \{1\}$$

ciii)  $f: (G = \{\pm 1, \pm i\}, \times) \rightarrow (Z_4, +_4)$

Defined by,  $f = \{(1, 0), (-1, 2), (i, 1), (-i, 3)\}$

$G = \{-i, -1, 1, i\}$   
 $Z_4 = \{[0], [1], [2], [3]\}$

$f(1) = 0$        $f(i) = 1$   
 $f(-1) = 2$       $f(-i) = 3$

→ For Homomorphism,

- We take,  $-i \in G, [1] \in Z_4$

$f(-i \times 1) = f(-i) = 3$  — ①

$f(-i) +_4 f(1) = 3$  — ②

From eq<sup>n</sup> 1 = 2

$f(-i \times 1) = f(-i) +_4 f(1)$

- We take,  $-1 \in G, [1] \in Z_4$

$f(-1 \times 1) = f(-1) = 2$  — ①

$f(-1) +_4 f(1) = 2$  — ②

From eq<sup>n</sup> 1 = 2

$f(-1 \times 1) = f(-1) +_4 f(1)$

- We take,  $1 \in G, [1] \in \mathbb{Z}_4$

$$F(1 \times 1) = F(1) = 0 \quad \text{--- (1)}$$

$$F(1) +_{\mathbb{Z}_4} F(1) = 0 \quad \text{--- (2)}$$

From eq<sup>n</sup> 1 = 2

$$\therefore F(1 \times 1) = F(1) +_{\mathbb{Z}_4} F(1)$$

- We take,  $i \in G, [1] \in \mathbb{Z}_4$

$$F(i \times 1) = F(i) = 1 \quad \text{--- (1)}$$

$$F(i) +_{\mathbb{Z}_4} F(1) = F(1) = 1 \quad \text{--- (2)}$$

From eq<sup>n</sup> 1 = 2

$$\therefore F(i \times 1) = F(i) +_{\mathbb{Z}_4} F(1)$$

Thus, the following Function is Homomorphism.

→ Kernel of Homomorphism:

Here, Identity element of  $\mathbb{Z}_4$  is 0 because the operation is addition.

$$\text{Ker}(F) = F(a) = 0$$

$$\therefore F(1) = 0$$

$$\therefore \text{Ker}(F) = \{1\}$$

Brain Spot



## \* Task 7: Rings, Integral Domain and Fields.

1 Define the following term with Properties and example.

(a) Ring: A non empty set  $R$  is under the binary operation addition and Multiplication, IF the following conditions are satisfied:

For  $(R, +, \cdot)$  is Ring, IF

(1)  $(R, +)$  is an abelian Group

(2)  $(R, \cdot)$  is a semigroup.

(3)  $R$  is follow Distributive Property,

$a, b, c \in R$ ,

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

(b) Integral Domain: A non empty set  $R$  is under the binary operation addition and Multiplication. IF the following conditions are satisfied:

For  $(R, +, \cdot)$  is Integral Domain,  
If,

- 1  $R$  is commutative.
- 2  $R$  has unit element.
- 3  $R$  is without zero divisors.

Ex. Ring of integers  $(\mathbb{Z}, +, \cdot)$  is an integral domain,

(C) Field: A Ring containing at least two elements is called a Field if,

- (i)  $R$  is commutative.
- (ii)  $R$  has unity.
- (iii)  $R$  is such that every non-zero element has multiplicative inverse in  $R$ .

Ex. A system  $(R, +, \cdot)$  is a Field if

- (i)  $(R, +)$  is an abelian Group.

cii)  $(R, 0)$  is a commutative Group.

ciii) The distributive laws,

$$a(b+c) = ab+ac, \quad \forall a, b, c \in R.$$

Brain Spot

\* Task: 8 : Introduction to Boolean Algebra.

Define the following term with example.

1 Boolean Algebra:

Boolean Algebra is the branch of algebra in which the values of the variables are the truth values, it usually denoted by 1 and 0.

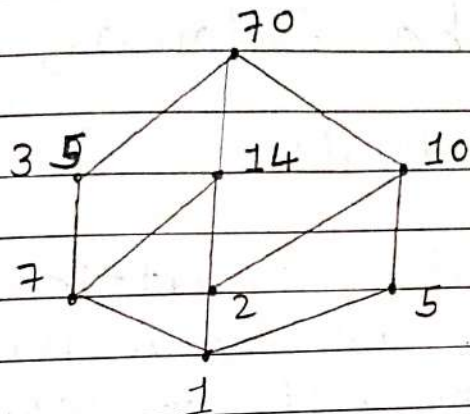
Ex.

a	b	$a \wedge b$
0	0	0
0	1	0
1	0	0
1	1	1

2 Sub Boolean Algebra:

Consider a boolean algebra  $(B, *, +, '0, 1)$  and Let  $A \subset B$  then  $(A, *, +, '0, 1)$  is called sub Boolean Algebra of B if A itself is a Boolean algebra, A contains the elements 0 and 1 and is closed under the operations  $*, +$  and  $'$ .

Ex. Consider the Boolean Algebra  $D_{70}$



### 3 Boolean Ring:

Boolean Ring  $R$  is a ring for where  $x^2 = x$  for all  $x$  in  $R$ , that is a ring that consists only of idempotent elements.

An example is the ring of integers modulo 2.

$$r^2 = r \text{ for } \forall r \in R.$$

\* Task: 9: Meet and Join.

1 Define the following terms.

(a) Join-irreducible:

If  $(B, *, \oplus)$  is lattice and  $a \in B$  is said to be Join-irreducible if it can not be express as a LUB of two distinct element of  $B$ .

(b) Meet-irreducible:

If  $(B, *, \oplus)$  is lattice and  $a \in B$  is said to be meet-irreducible if it can not be express as a GLB of two distinct element of  $B$ .

(c) Atoms:

Let  $(B, *, \oplus)$  be a lattice and  $a \in B$  then  $a$  is said to be Atom if it satisfy following Property.

(i)  $a$  is Join-irreducible.

(ii)  $a$  is cover of  $0$ .

(c) Anti-atoms:

Let  $(B, *, \oplus)$  be a lattice and a GB then  $a$  is said to be anti-atoms if it satisfy following Property.

(i)  $a$  is Meet-irreducible.

(ii)  $a$  is cover of 1.

2 Prove the following Boolean Identities.

(a)  $a \oplus (a' * b) = a \oplus b$

$$\begin{aligned} \text{L.H.S.} &= a \oplus (a' * b) \\ &= (a \oplus a') * (a \oplus b) \\ &= 1 * (a \oplus b) \\ &= a \oplus b \end{aligned}$$

L.H.S. = R.H.S.

(b)  $a * (a' \oplus b) = a * b$

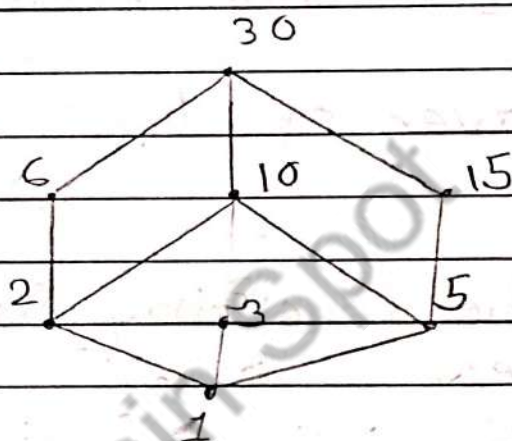
$$\begin{aligned} \text{L.H.S.} &= a * (a' \oplus b) \\ &= (a * a') \oplus (a * b) \\ &= 0 \oplus (a * b) \\ &= a * b \end{aligned}$$

L.H.S. = R.H.S.

3 Find Join-irreducible, Meet-irreducible atoms and antiatoms of following lattices.

(a)  $\langle S_{30}, \mathcal{D} \rangle$

$$S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$$



Join-irreducible - 2, 3, 5

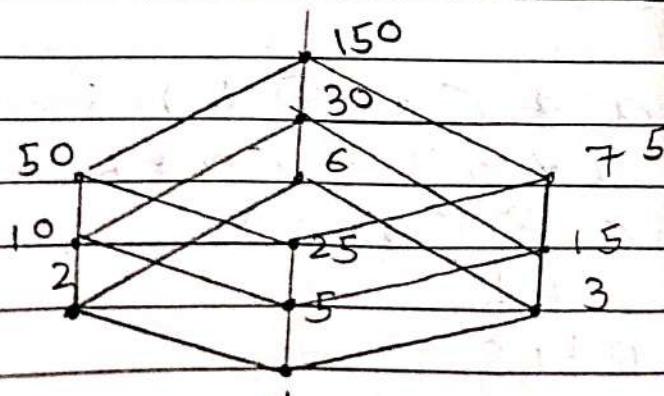
Meet-irreducible - 6, 10, 15

Atoms - 2, 3, 5

Anti-atoms - 6, 10, 15

(b)  $\langle S_{150}, \mathcal{D} \rangle$

$$S_{150} = \{1, 2, 3, 5, 10, 6, 15, 25, 50, 30, 75, 150\}$$





Join-irreducible - 2, 3, 5, 25

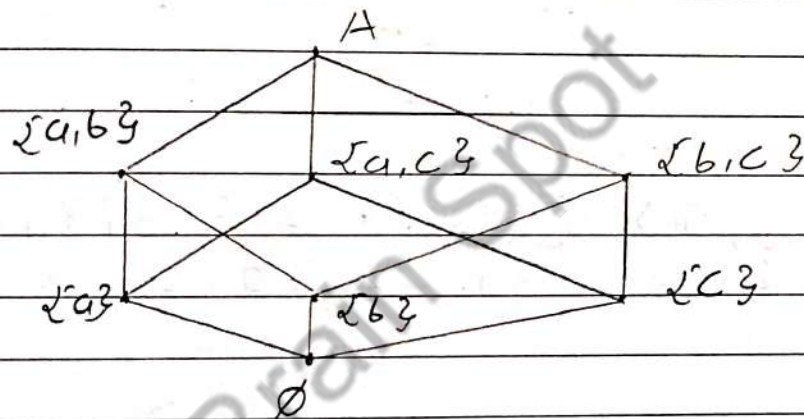
Atoms - 2, 3, 5

Meet-irreducible - 6, 30, 50, 75

Anti-atoms - 30, 50, 75

(c)  $\langle \text{PCA}, \subseteq \rangle$  where  $A = \{a, b, c\}$

$\text{PCA} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \emptyset, A\}$



Join-irreducible -  $\{a\}, \{b\}, \{c\}$

Atoms -  $\{a\}, \{b\}, \{c\}$

Meet-irreducible -  $\{a, b\}, \{a, c\}, \{b, c\}$

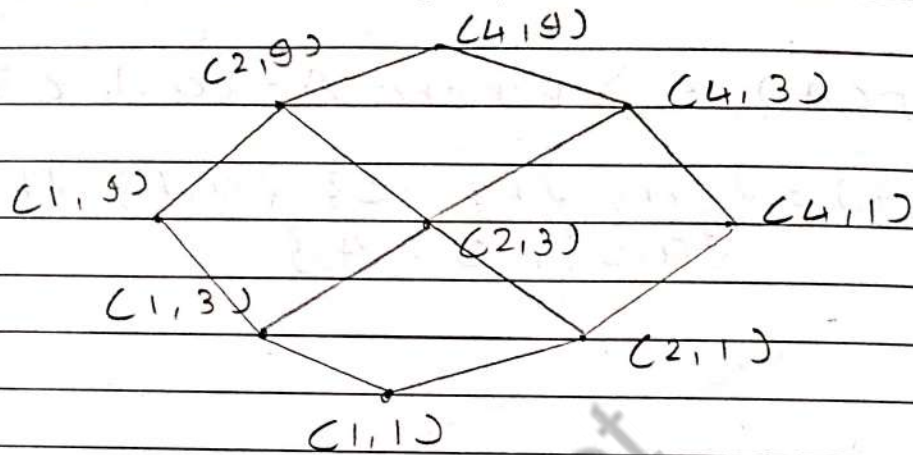
Anti-atoms -  $\{a, b\}, \{a, c\}, \{b, c\}$

(d)  $\langle S_4 \times S_9, \mathcal{D} \rangle$

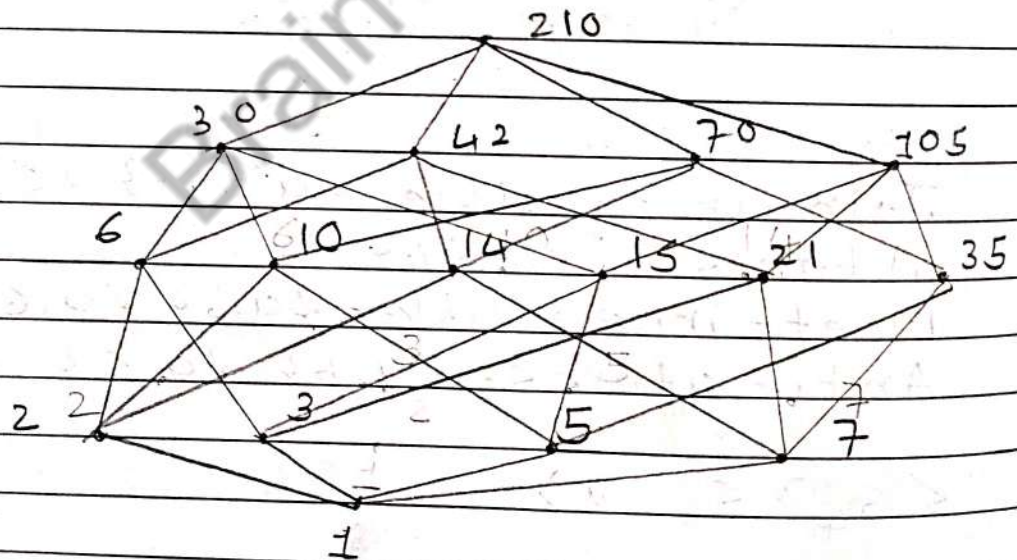
$S_4 = \{1, 2, 4\}, S_9 = \{1, 3, 9\}$

$S_4 \times S_9 = \{(1, 1), (1, 3), (2, 1), (4, 1), (1, 9), (2, 3), (2, 9), (4, 3), (4, 9)\}$

Join-irreducible -  $(1,3), (1,9), (2,1), (4,1)$   
 Meet-irreducible -  $(2,9), (4,3), (4,1), (1,9)$   
 Atoms -  $(1,3), (2,1)$   
 Antiatoms -  $(2,9), (4,3)$



(c)  $\langle S_{210}, 0 \rangle$   $S_{210} = \{1, 2, 3, 5, 7, 6, 10, 14, 15, 21, 35, 30, 42, 70, 210\}$



Join-irreducible -  $2, 3, 5, 7$   
 Meet-irreducible -  $30, 42, 70, 105$   
 Atoms -  $2, 3, 5, 7$   
 Antiatoms -  $30, 42, 70, 105$

\* Task: <sup>10</sup> Boolean Expression,  
SOP or POS.

1 Find the SOP and POS Expansions  
of the following Boolean Function.

(a)  $F(x, y, z) = x + y + z$

x	y	z	$x + y + z$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

$$F(x, y, z) = \sum(1, 2, 3, 4, 5, 6, 7)$$

$$= \Pi(0)$$

(b)  $F(x, y, z) = (x + z)y$

x	y	z	$(x + z)y$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0

1	1	0	1
1	1	1	1

$$F(x, y, z) = \Sigma(3, 6, 7)$$

$$= \Pi(0, 1, 2, 4, 5)$$

c.c)  $F(x, y, z) = x + (yz)'$

x	y	z	(yz)'	x + (yz)'
0	0	0	1	1
0	0	1	1	1
0	1	0	1	1
0	1	1	0	0
1	0	0	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

$$F(x, y, z) = \Sigma(0, 1, 2, 4, 5, 6, 7)$$

$$= \Pi(3)$$

c.d)  $F(x, y, z) = (x + y)' + (x'z)$

x	y	z	(x+y)'	(x'z)	(x+y)' + (x'z)
0	0	0	1	0	1
0	0	1	1	1	1
0	1	0	0	0	0
0	1	1	0	1	1
1	0	0	0	0	0
1	0	1	0	0	0
1	1	0	0	0	0

1 | 1 | 1 | 0 | 0 | 0

$$F(x, y, z) = \sum (0, 1, 3) \\ = \prod (2, 4, 5, 6, 7)$$

(e)  $F(x, y, z, w) = (xy'z) + w$

x	y	z	w	xy'	xy' + w
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	0	0
0	0	1	1	0	1
0	1	0	0	0	0
0	1	0	1	0	1
0	1	1	0	0	0
0	1	1	1	0	1
1	0	0	0	1	1
1	0	0	1	1	1
1	0	1	0	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	0	1	0	1
1	1	1	0	0	0
1	1	1	1	0	1

$$F(x, y, z, w) = \sum (1, 3, 5, 7, 8, 9, 10, 11, 13, 15)$$

$$= \prod (0, 2, 4, 6, 12, 14)$$

\* Task: 1.1: Boolean Expression and Equivalence.

1 In any Boolean algebra, show that,

(a)  $a = b \Leftrightarrow ab' + a'b = 0$

Let  $a = b = c$

$$\begin{aligned} \text{L.H.S.} &= ab' + a'b \\ &= cc' + c'c \\ &= 0 \end{aligned}$$

L.H.S. = R.H.S.

(b)  $a = 0 \Leftrightarrow ab' + a'b = b$

$$\begin{aligned} \Rightarrow \text{L.H.S.} &= ab' + a'b \\ &= 0 + b \\ &= b \end{aligned}$$

L.H.S. = R.H.S.

$$\Rightarrow ab' + a'b = b$$

If  $a = 1$  then  $b' + 0 = b$  not possible

If  $a = 0$  then  $0 + b = b$  Possible.

$$c) (a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a)$$

$$\begin{aligned} \Rightarrow \text{L.H.S} &= (a+b')(b+c')(c+a') \\ &= (ab + ac' + bb' + b'c') \cdot (c+a') \\ &= abc + ac'c + bb'c + aa'b \\ &\quad + aa'c + bb'a' + b'c'a' \\ &= abc + b'a'a' \\ &= \text{R.H.S} \end{aligned}$$

$$\begin{aligned} \Rightarrow \text{R.H.S} &= (a'+b)(b'+c)(c'+a) \\ &= (a'b' + a'c + bb' + bc)(c'+a) \\ &= a'b'c' + a'cc' + bb'c' + bcc' \\ &\quad + aa'b + aa'c + bb'a + abc \\ &= abc + a'b'c' \end{aligned}$$

$$\therefore \text{L.H.S} = \text{R.H.S}$$

2 Simplify the following Boolean Expressions.

$$ca) (a * b') \oplus (a \oplus b)'$$

$$= (a' \oplus b') \oplus (a' * b')$$

$$= ((a' \oplus b') \oplus a') * ((a' \oplus b') \oplus b')$$

$$= ((a' \oplus a') \oplus b') * ((a' \oplus b') \oplus b')$$

$$= (a' \oplus b') * (a' \oplus b')$$

$$= a' \oplus b'$$

$$cb) (a' * b' * c) \oplus (a * b' * c) \oplus (a * b' * c')$$

$$= \{ (a' \oplus a) * (b' \oplus a) * (c \oplus a) \} *$$

$$\{ (a' \oplus b') * (b' \oplus b') * (c \oplus b') \} *$$

$$\{ (a' \oplus c) * (b' \oplus c) * (c \oplus c) \}$$

$$\oplus (a * b' * c')$$

$$= (b' \oplus a) * (c \oplus a) * (a' \oplus b') * b' * (c \oplus b)$$