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Chapter - 4 Beta and Gamma Function

* Task - 1 Evaluate the following integrals

1 $\int_{-\infty}^{\infty} e^{-4x^2} \cdot dx$

take $4x^2 = t \rightarrow x = \frac{\sqrt{t}}{2}$

$\therefore 8x \cdot dx = dt$

$\therefore dx = \frac{dt}{8x} = \frac{\sqrt{t}}{4} \cdot dt$

$= \int_{-\infty}^{\infty} e^{-t} \cdot \frac{\sqrt{t}}{4} dt$

$= \int_{-\infty}^{\infty} e^{-t} \cdot \frac{t^{\frac{1}{2}}}{4} \cdot dt$

$= \frac{1}{4} \int_{-\infty}^{\infty} e^{-t} \cdot t^{1-\frac{1}{2}} \cdot dt$

$= \frac{\sqrt{1/2}}{4}$

$= \frac{\sqrt{\pi}}{4}$

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2 $\int_0^{\infty} x^2 \cdot e^{-4x^4} \cdot dx$

take $x^4 = t$

$\therefore 4x^3 \cdot dx = dt$

$\therefore dx = \frac{dt}{4x^3}$

$\therefore x = \sqrt[4]{t}$

$\therefore x^2 = t^{1/2}$

$\therefore x^3 = t^{3/4}$

$= \int_0^{\infty} x^2 \cdot e^{-t} \cdot \frac{dt}{4x^3}$

$\therefore \int_0^{\infty} \cdot e^{t} \cdot \frac{dt}{4 \cdot t^{1/4}}$

$\therefore \frac{1}{4} \int_0^{\infty} e^{-t} \cdot t^{-\frac{1}{4}} \cdot dt$

$= \frac{1}{4} \int_0^{\infty} e^{-t} \cdot t^{\frac{3}{4} - 1} \cdot dt$

$= \frac{1}{4} \Gamma 3/4$

3 $$\int_0^1 \sqrt{x}\,(1-x^2)^{\frac{1}{3}} \cdot dx$$

take $x^2 = t \rightarrow x = \sqrt{t}$

$$\therefore 2x \cdot dx = dt$$

$$\therefore dx = \frac{dt}{2x}$$

$$\therefore dx = \frac{dt}{2\sqrt{t}}$$

$$= \int_0^1 t^{\frac{1}{4}} (1-t)^{\frac{1}{3}} \cdot \frac{dt}{2t^{1/2}}$$

$$= \frac{1}{2} \int_0^1 t^{-\frac{1}{4}} \cdot (1-t)^{\frac{1}{3}} \cdot dt$$

$$= \frac{1}{2} \int_0^1 t^{\frac{3}{4}-1} \cdot (1-t)^{\frac{4}{3}-1} \cdot dt$$

$$= \frac{1}{2} \beta\left(\frac{3}{4}, \frac{4}{3}\right)$$

4 $$\int_0^2 x^4 (8-x^3)^{-\frac{1}{3}} \cdot dx$$

take $x^3 = 8t$

$$\therefore 3x^2 \cdot dx = 8 \cdot dt$$

$$\therefore dx = \frac{8}{3} \cdot \frac{dt}{x^2}$$

$$\therefore dx = \frac{dt}{3} \times \frac{8}{\frac{4}{2} \cdot t^{2/3}}$$

$$= \int_0^1 16 \cdot t^{4/3} \cdot (8 - 8t)^{-1/3} \cdot \frac{dt}{6}$$

$$= \int_0^1 \frac{16 \times 2}{6} \cdot t^{4/3} \cdot (1-t)^{-1/3} \cdot dt$$

$$= \frac{16}{3} \int_0^1 (1-t)^{\frac{2}{3} - 1} \cdot t^{\frac{5}{3} - 1} \cdot dt$$

$$= \frac{16}{3} \beta\left(\frac{5}{3}, \frac{2}{3}\right)$$

* Task-2 Evaluate the following integrals.

1 $\int_0^{\pi} \sin^2 x (1+\cos x)^4 \cdot dx$

$= \int_0^{\pi} \sin^2 x \left(2\cos^2 \frac{x}{2}\right)^4 \cdot dx$

$= 16 \int_0^{\pi} \sin^2 x \cdot \cos^8 \frac{x}{2} \cdot dx$

Here, we take $\frac{x}{2} = t$

$\therefore \frac{1}{2} \cdot dx = dt$

$\therefore dx = 2 \cdot dt$

$\therefore x = 2t$

When $x = \pi$, $\frac{x}{2} \rightarrow \frac{\pi}{2}$
$x = 0$, $\frac{x}{2} \rightarrow 0$

$= 16 \int_0^{\pi/2} \sin^2 2t \cdot \cos^8 x \cdot 2 \cdot dt$

$= 16 \int_0^{\pi/2} 2\sin^2 t \cdot \cos^2 t \cdot \cos^8 x \cdot 2 \cdot dt$

$$= 64 \int_0^{\pi/2} \sin\theta \cdot \cos^9\theta \cdot d\theta$$

$$= 64 \cdot \frac{1}{2} \beta\left(\frac{1+1}{2}, \frac{9+1}{2}\right) \cdot d\theta$$

$$= 32 \frac{\Gamma 1 \; \Gamma 5}{\Gamma 1+5}$$

$$= 32$$

$$= 64 \int_0^{\pi/2} 2 \sin^2\theta \cdot \cos^{10}\theta \cdot d\theta$$

$$= 128 \int_0^{\pi/2} \sin^2\theta \cdot \cos^{10}\theta \cdot d\theta$$

$$= 128 \frac{\beta}{2}\left(\frac{2+1}{2}, \frac{10+1}{2}\right)$$

$$= 64 \frac{\Gamma 3/2 \cdot \Gamma 11/2}{\Gamma \frac{3}{2}+\frac{11}{2}}$$

$$= \frac{64 \times 1/2 \times \Gamma 1/2 \times 9/2 \times 7/2 \times 5/2 \times 3/2 \times 1/2 \times \Gamma 1/2}{\Gamma 7}$$

$$= \frac{21\pi}{16}$$

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2 $\int_0^{\pi/2} \sin 3x \cdot \cos^5 x \cdot dx$

$$= \int_0^{\pi/2} (3\sin x - 4\sin^3 x) \cdot \cos^5 x \cdot dx$$

$$= \int_0^{\pi/2} 3\sin x \cdot \cos^5 x \cdot dx - 4\int_0^{\pi/2} \sin^3 x \cdot \cos^5 x \cdot dx$$

$$= 3 \frac{\beta}{2}\left(\frac{1+1}{2}, \frac{5+1}{2}\right) - 4 \frac{\beta}{2}\left(\frac{3+1}{2}, \frac{5+1}{2}\right)$$

$$= \frac{3}{2} \frac{\boxed{1} \cdot \boxed{3}}{\boxed{1+3}} - \frac{4}{2} \frac{\boxed{2} \cdot \boxed{3}}{\boxed{2+3}}$$

$$= \frac{3 \times 2 \times 1}{2 \times 3 \times 2} - \frac{2 \times 2 \times 1}{4 \times 3 \times 2}$$

$$= \frac{1}{2} - \frac{1}{6}$$

$$= \frac{1}{3}$$

3 $\displaystyle\int_0^1 \underset{v}{x^5} \cdot \underset{u}{\sin^{-1}x} \cdot dx$

$$= \sin^{-1}x \int_0^1 x^5 \cdot dx - \int_0^1 \frac{d}{dx}\sin^{-1}x \cdot \int x^5 \cdot dx$$

$$= \left[\sin^{-1}x \left[\frac{x^6}{6}\right]\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^6}{6} \cdot dx$$

$$= \frac{\pi}{2} \cdot \frac{1}{6} - 6\int_0^1 \frac{1 \cdot x^6}{\sqrt{1-x^2}} \cdot dx$$

Here we take $x = \sin\theta$

$\therefore dx = \cos\theta \cdot d\theta$

x	0	1
θ	0	$\pi/2$

$$= \frac{\pi}{12} - 6\int_0^{\frac{\pi}{2}} \frac{\sin^6\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta \cdot d\theta$$

$$= \frac{\pi}{12} - 6\int_0^{\frac{\pi}{2}} \frac{\sin^6\theta}{\cos\theta} \cdot \cos\theta \cdot d\theta$$

$$= \frac{\pi}{12} - 6\int_0^{\frac{\pi}{2}} \sin^6\theta \cdot \cos^0\theta \cdot d\theta$$

$$= \frac{\pi}{12} - \frac{6\beta}{2}\left(\frac{6+1}{2}, \frac{1}{2}\right)$$

$$= \frac{\pi}{12} - \frac{3\,\Gamma(7/2)\cdot\Gamma(1/2)}{\Gamma\left(\frac{7}{2}+\frac{1}{2}\right)}$$

$$= \frac{\pi}{12} - \frac{3 \times 5/2 \times 3/2 \times 1/2 \times \Gamma(1/2) \times \Gamma(1/2)}{\Gamma 4}$$

$$= \frac{\pi}{12} - \frac{5\pi}{32 \times 6}$$

$$= \frac{11\pi}{192}$$

4 $$\int_0^1 \frac{x^7}{\sqrt{1-x^2}}\,dx$$

Here we take $x = \sin\theta$

$\therefore dx = \cos\theta \cdot d\theta$

x	0	1
θ	0	$\pi/2$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^7\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta \cdot d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^7\theta \cdot \cos\theta \cdot d\theta}{\cos\theta}$$

$$= \int_0^{\pi/2} \sin^7\theta \cdot \cos^0\theta \cdot d\theta$$

$$= \frac{\beta}{2}\left(\frac{7+1}{2}, \frac{1}{2}\right)$$

$$= \frac{\beta}{2}\left(4, \tfrac{1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma 4 \; \Gamma \tfrac{1}{2}}{\Gamma\left(4+\tfrac{1}{2}\right)}$$

$$= \frac{1 \times 2 \times \Gamma \tfrac{1}{2}}{2 \times \tfrac{7}{2} \times \tfrac{5}{2} \times \tfrac{3}{2} \times \tfrac{1}{2} \times \Gamma \tfrac{1}{2}}$$

$$= \frac{16}{35}$$

5. $$\int_0^{\pi} (1 - \cos\theta)^5 \cdot d\theta$$

We know that, $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$

$$= \int_0^{\pi} \left(2\sin^2\frac{\theta}{2}\right)^5 \cdot d\theta$$

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$$= 32\int_0^{\pi} \sin^{10}\frac{\theta}{2} \cdot d\theta$$

Here we take $\frac{\theta}{2} = t$

$\therefore d\theta = 2 \cdot dt$

θ	0	π
t	0	$\pi/2$

$$= 32\int_0^{\pi/2} \sin^{10}t \cdot 2 \cdot dt$$

$$= 64 \cdot \frac{\beta}{2}\left(\frac{10+1}{2}, \frac{1}{2}\right)$$

$$= 32 \cdot \frac{\Gamma 11/2 \cdot \Gamma 1/2}{\Gamma \frac{11}{2}+\frac{1}{2}}$$

$$= \frac{32 \times 9/2 \times 7/2 \times 5/2 \times 3/2 \times 1/2 \times \Gamma 1/2 \times \Gamma 1/2}{5 \times 4 \times 3 \times 2}$$

$$= \frac{63\pi}{8}$$

6 $$\int_{0}^{\infty} \frac{1}{(1+x^2)^{9/2}} \cdot dx$$

Here we take $x = \tan\theta$

$$\therefore d\theta = \sec^2\theta \cdot d\theta$$

x	0	∞
θ	0	$\pi/2$

$$= \int_{0}^{\frac{\pi}{2}} \frac{1}{(1+\tan^2\theta)^{9/2}} \cdot \sec^2\theta \cdot d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{1}{(\sec^2\theta)^{9/2}} \cdot \sec^2\theta \cdot d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{\sec^2\theta}{\sec^9\theta} \cdot d\theta$$

$$= \int_{0}^{\pi/2} \frac{1}{\sec^7\theta} \cdot d\theta$$

$$= \int_{0}^{\pi/2} \cos^7\theta \cdot d\theta$$

$$= \frac{\beta}{2}\left(\frac{1}{2}, \frac{7+1}{2}\right)$$

$$= \frac{\beta}{2}\left(\frac{1}{2}, 4\right)$$

$$= \frac{1}{2}\,\frac{\overline{|1/2}\,\overline{|4}}{\overline{|\frac{1}{2}+4}}$$

$$= \frac{1}{2} \cdot \frac{\times \overline{|1/2} \times 2}{\times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \overline{|1/2}}$$

$$= \frac{16}{35}$$

7 $$\int_0^{\infty} \frac{1}{(1+x^2)^5} \cdot dx$$

Here we take $x = \tan\theta$

$\therefore dx = \sec^2\theta \cdot d\theta$

x	0	∞
θ	0	$\pi/2$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{(1+\tan^2\theta)^5} \cdot \sec^2\theta \cdot d\theta$$

$$= \int_0^{\pi/2} \frac{\sec^2\theta}{\sec^{10}\theta} \cdot d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^8\theta \cdot d\theta$$

$$= \frac{\beta}{2}\left(\frac{1+0}{2}, \frac{8+1}{2}\right)$$

$$= \frac{1}{2} \times \frac{\Gamma\tfrac{1}{2} \times \tfrac{7}{2} \times \tfrac{5}{2} \times \tfrac{3}{2} \times \tfrac{1}{2} \times \Gamma\tfrac{1}{2}}{4 \times 3 \times 2}$$

$$= \frac{35\pi}{256}$$

8 $$\int_0^\infty \frac{x^2}{(1+x^6)^{7/2}} \cdot dx$$

Here we take $x^3 = \tan\theta$

$\therefore 3x^2 \, dx = \sec^2\theta \cdot d\theta$

x	0	∞
θ	0	$\pi/2$

$$= \int_0^{\frac{\pi}{2}} \frac{(\tan^2\theta)^{1/3}}{(1+\tan^2\theta)^{7/2}} \cdot \frac{\sec^2\theta}{3(\tan^2\theta)^{1/3}} \cdot d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3} \cos^5\theta \cdot d\theta$$

$$= \frac{1}{3} \cdot \frac{\beta}{2}\left(\frac{1}{2}, \frac{5+1}{2}\right)$$

$$= \frac{1}{6} \times \frac{\left\lceil 1/2 \cdot \right\lceil 3}{\left\lceil \frac{1}{2}+3 \right.}$$

$$= \frac{1 \cdot \left\lceil 1/2 \cdot 2\right.}{6 \times 5/2 \times 3/2 \times 1/2 \times \left\lceil 1/2\right.}$$

$$= \frac{8}{45}$$

9 $\int_0^{\pi} (1+\cos\theta)^3 \cdot d\theta$

We know that $1+\cos\theta = 2\cos^2\frac{\theta}{2}$

$$= \int_0^{\pi} 8\cos^6\frac{\theta}{2} \cdot d\theta$$

Here we take $\frac{\theta}{2} = t$

$\therefore d\theta = 2 \cdot dt$

θ	0	π
t	0	$\pi/2$

$$= 8\int_0^{\pi/2} \cos^6 t \cdot 2 \cdot dt$$

$$= 16\int_0^{\pi/2} \cos^6 t \cdot dt$$

$$= \frac{16 \cdot \beta}{2}\left(\frac{1}{2}, \frac{6+1}{2}\right)$$

$$= \frac{8 \cdot \Gamma 1/2 \cdot \Gamma 7/2}{\Gamma \frac{1}{2}+\frac{7}{2}}$$

$$= \frac{8 \cdot \Gamma 1/2 \cdot 5/2 \times 3/2 \times 1/2 \times \Gamma 1/2}{3 \times 2 \times 1}$$

$$= \frac{5\pi}{2}$$

10 $\int_0^{\frac{\pi}{4}} (\cos^7 2\theta) \cdot d\theta$

Here we take $2\theta = t$

$\therefore 2 \cdot d\theta = dt$

$\therefore d\theta = \frac{dt}{2}$

θ	0	0
t	$\pi/4$	$\pi/2$

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$$= \int_0^{\frac{\pi}{2}} \cos^7\theta \cdot \frac{dt}{2}$$

$$= \frac{1}{2} \cdot \frac{\beta}{2}\left(\frac{1}{2}, \frac{7+1}{2}\right)$$

$$= \frac{1}{4} \cdot \frac{\sqrt{1/2}\sqrt{4}}{\sqrt{\frac{1}{2}+4}}$$

$$= \frac{1 \times \sqrt{1/2} \times 3 \times 2}{4 \times 7/2 \times 5/2 \times 3/2 \times 1/2 \times \sqrt{1/2}}$$

$$= \frac{8}{35}$$