

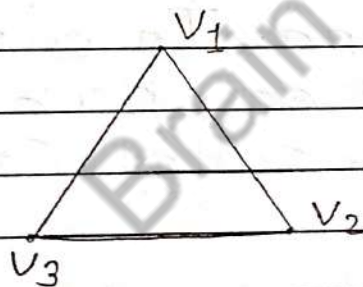
Unit - 5 Graph Theory

* Task : 1 : Basic Concepts of Graphs and Diagraphs.

1 Define the following terms of the Graph with example.

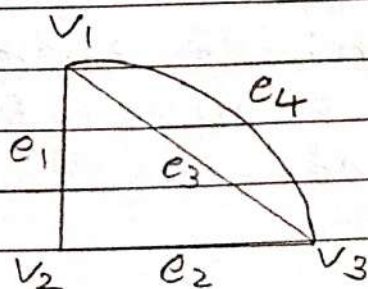
(i) Simple Graph: A Graph which has neither loop and multiple edges is called Simple Graph.

Ex.



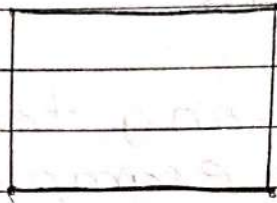
(ii) Multiple Graph: A Graph in which no loop are allowed but more than one edges are join two vertex.

Ex.



ciii) **Regular Graph:** In a Graph, every vertex has same degree and same edges this Graph is called Regular Graph.

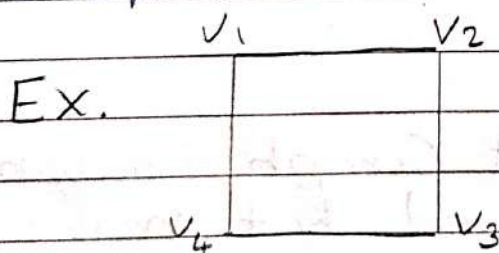
Ex.



2-Regular
Graph

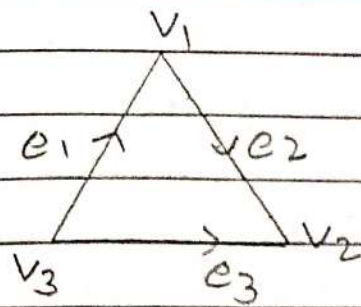
civ) **Null Graph:** In a Graph, edges set is empty then this Graph is said to be null Graph.

cv) **Complete Graph:** In a Graph, edges have every pair of vertex, then graph is called Complete Graph.



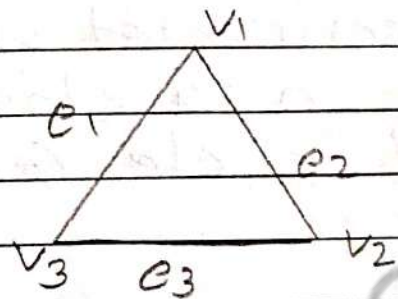
cvi) **Directed Graph:** If every edges of Graph G is directed then, Graph is called Directed Graph.

Ex



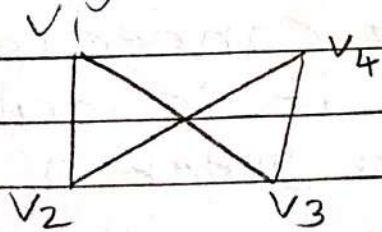
(vii) Undirected Graph: If every edges of Graph G is not directed then this Graph is called Undirected Graph.

Ex.



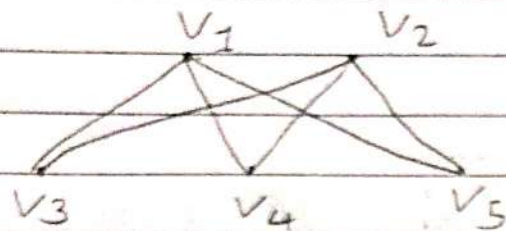
(viii) Bipartite Graph: A set of graph vertices decomposed into two disjoint set such that no graph vertices within the same set are adjacent.

Ex.

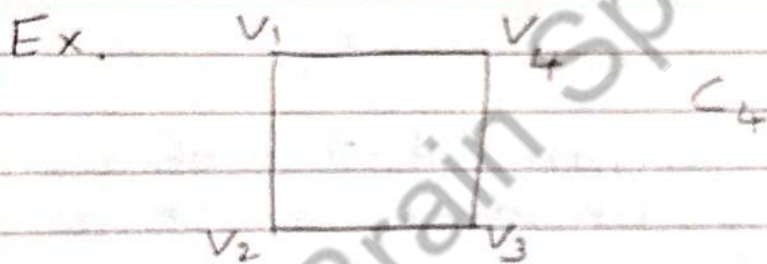


(ix) Complete Bipartite Graph: In a Bipartite Graph in which every vertex v_1 and is joining each vertex v_2 by one edges.

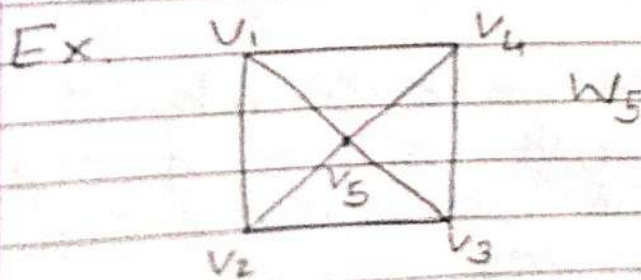
Ex. $K_{2,3}$



(x) Cycle Graph: In a Graph, If every edges connected with a every vertex in a closed chain This is called cycle Graph.



(xi) Wheel Graph: In a cycle Graph, every vertex is connected with a new vertex it is called wheel Graph of order n.



(xii) **Pendant Vertex:** A vertex with degree with one is called Pendant vertex

Ex. 

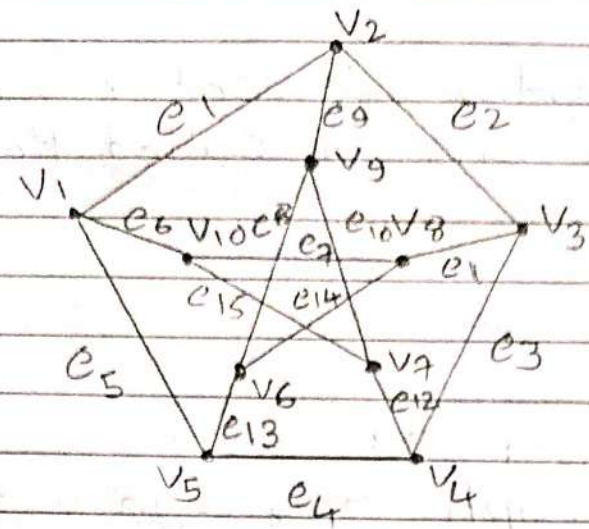
(xiii) **Labelled Graph:** A graph with each node labelled differently, so that all nodes are considered distinct for purposes.

2 State Handshaking lemma and verify it for the graph.

In any undirected Graph G , number of vertex V and number of edges are E then,

$$\sum_{i=1}^n d(v_i) = 2E$$

(a)



- degree $(V_1) = 3$
- degree $(V_2) = 3$
- degree $(V_3) = 3$
- degree $(V_4) = 3$
- degree $(V_5) = 3$
- degree $(V_6) = 3$
- degree $(V_7) = 3$
- degree $(V_8) = 3$
- degree $(V_9) = 3$
- degree $(V_{10}) = 3$

$$\sum_{i=1}^{10} d(V_i) = 30 \quad \text{--- (1)}$$

By them,

$$\sum_{i=1}^{10} d(V_i) = 2e$$

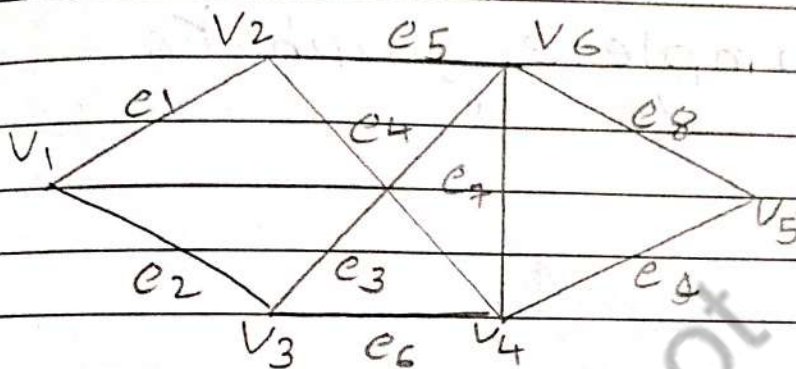
$$= 2 \times 15$$

$$= 30 \quad \text{--- (2)}$$

$$ed^n 1 = 2$$

So, Hand Shaking Lem
Lemma is prove.

(b)



$$\text{degree}(V_1) = 2$$

$$\text{degree}(V_2) = 3$$

$$\text{degree}(V_3) = 3$$

$$\text{degree}(V_4) = 3$$

$$\text{degree}(V_5) = 2$$

$$\text{degree}(V_6) = 4$$

$$\sum_{i=1}^6 d(V_i) = 18 - \textcircled{1}$$

By thm,

$$\sum_{i=1}^6 d(V_i) = 2e$$

$$= 2 \times 9$$

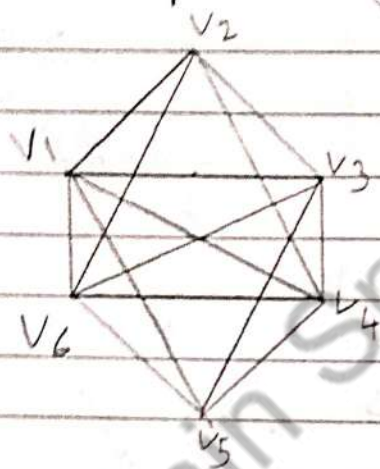
$$= 18 - \textcircled{2}$$

$$e_4^n = 1 = 2.$$

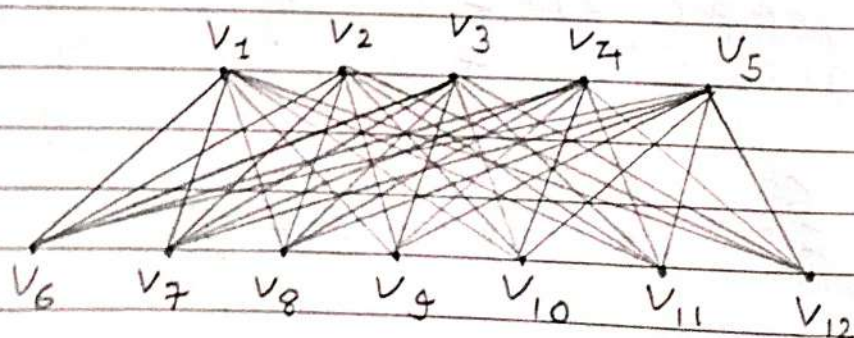
So, Handshaking lemma is proved.

3 Draw the following Graph:

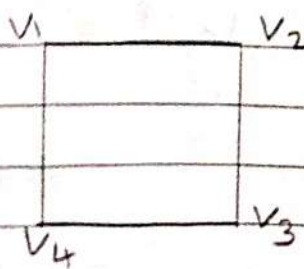
(i) The complete graph K_6



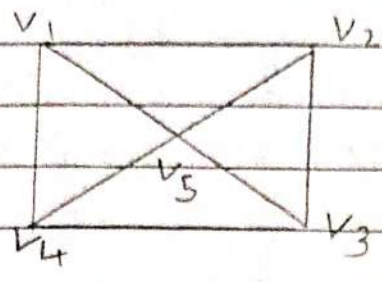
(ii) Complete Bipartite Graph $K_{5,7}$



(iii) Cycle Graph C_4 :

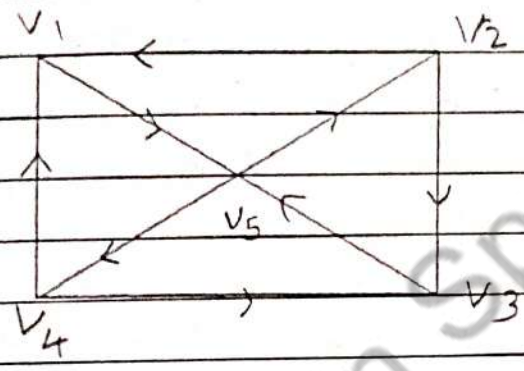


(iv) Wheel Graph W_5 :



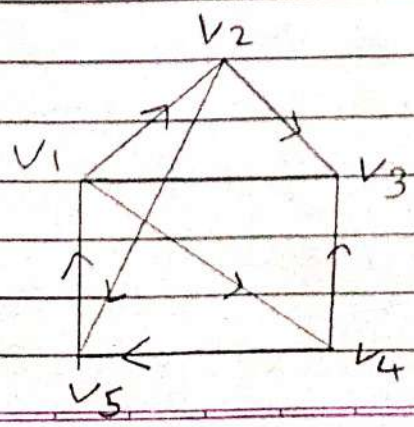
4 Find out degree and in degree.

(a)



Vertex	In-degree	Out-degree
V_1	2	1
V_2	1	2
V_3	2	1
V_4	1	2
V_5	2	2

(b)



Vertex	In-degree	Out-degree
V_1	1	2
V_2	1	2
V_3	2	0
V_4	1	2
V_5	2	1

Brain Spot

* Task : 5 : Matrix Representation of Graphs and Diagraphs.

1 Define the following terms of undirected and directed graph with example.

(i) Adjacency Matrix:

→ Undirected Graph:

Let $G = (V, E)$ be the graph, where $V = \{v_1, v_2, \dots\}$ be the vertex set then adjacency matrix of graph is an $n \times n$ matrix $A = [a_{ij}]$

$$a_{ij} = \begin{cases} 1 & \text{- IF } v_i \text{ and } v_j \text{ is connected.} \\ 0 & \text{- IF } v_i \text{ and } v_j \text{ not connected.} \end{cases}$$

→ Directed Graph:

Let $G = (V, E)$ be the graph, where $V = \{v_1, v_2, \dots\}$ be the vertex set then adjacency matrix of graph is an $n \times n$ matrix $A = [a_{ij}]$

$$a_{ij} = \begin{cases} 1 & \text{- IF } v_i \text{ from to } v_j \\ 0 & \text{- IF } v_i \text{ is not from } v_j \end{cases}$$

cii) Path Matrix: Let $G = (V, E)$ is any graph and V_1 and V_2 any two vertex of graph G , if there are p path between vertices, V_1 and V_2 then it is called path matrix.

$$P_{ij} = \begin{cases} 1 & \text{if edges } e_j \text{ in} \\ & \text{the path} \\ 0 & \text{if edges } e_j \text{ is} \\ & \text{not in path} \end{cases}$$

ciii) Incident Matrix:

→ Undirected Graph:

Let $G = (V, E)$ be the graph where $V = \{V_1, V_2, \dots\}$ be a vertex set and edge set then incident matrix is $n \times n$ Matrix $A = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if the } V_i \text{ is} \\ & \text{incident of } e_j \\ 0 & \text{otherwise} \end{cases}$$

→ Directed Graph:

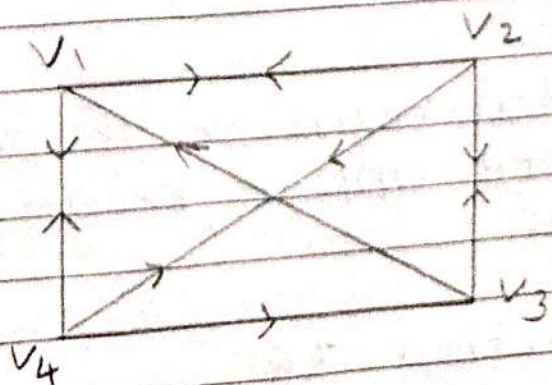
Let $G = (V, E)$ be the graph where $V = \{v_1, v_2, \dots\}$ be a vertex set and edges set then incident matrix $A = [a_{ij}]$ is defined by.

$$a_{ij} = \begin{cases} 1 & \text{- IF } v_i \text{ is initial of} \\ & \text{vertices edges } e_j \\ -1 & \text{- IF } v_i \text{ is end of} \\ & \text{edges } e_j \\ 0 & \text{- Not incident.} \end{cases}$$

2 Let the adjacency matrix of a graph $G = (V, E)$ be

	v_1	v_2	v_3	v_4
v_1	0	1	0	1
v_2	1	0	1	1
v_3	1	1	0	0
v_4	1	1	1	0

(a) Out



c) Out degree of V_2

$$\text{Out degree of } V_2 = 3$$

c) The number of paths From V_1 to V_2

$$1) V_1, V_2, V_3$$

$$2) V_1, V_2, V_3, V_4$$

$$3) V_1, V_2, V_4$$

c) Total Number of paths of length 2

For total number of paths,

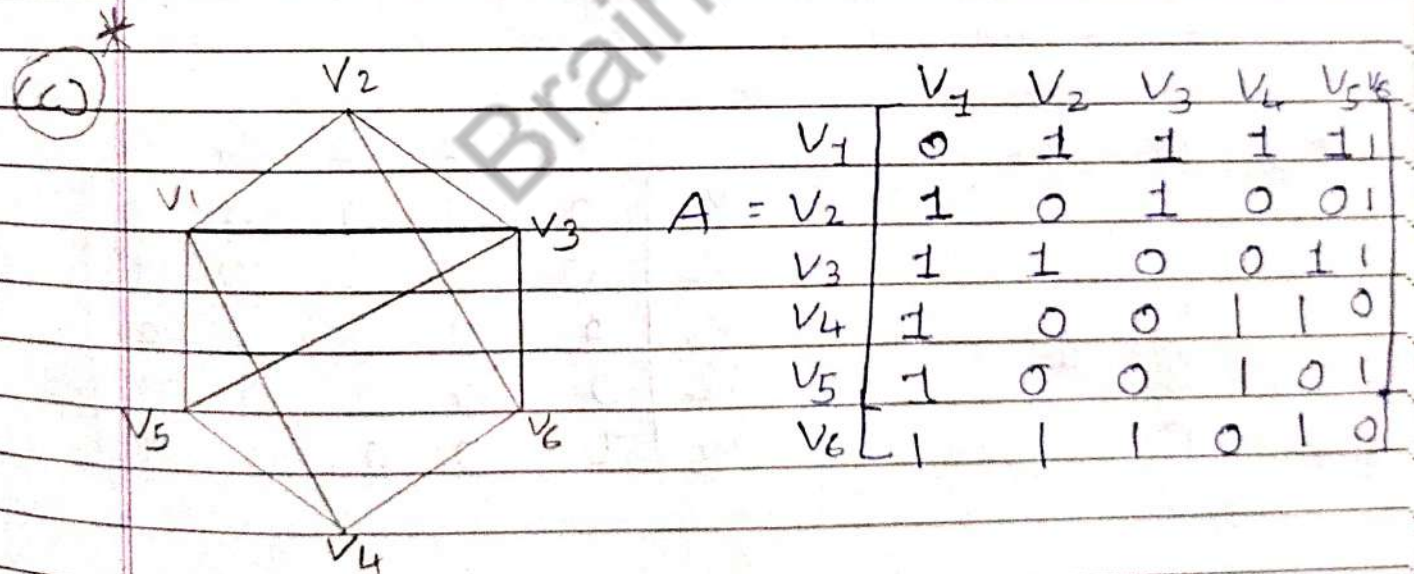
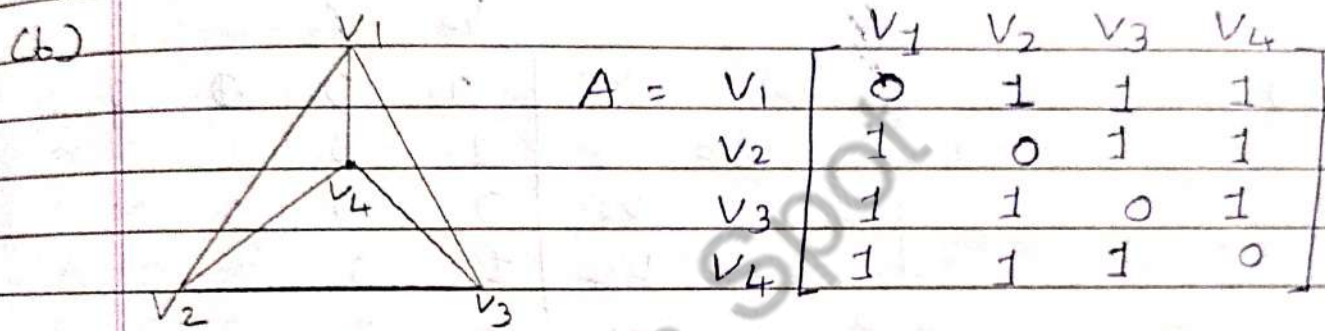
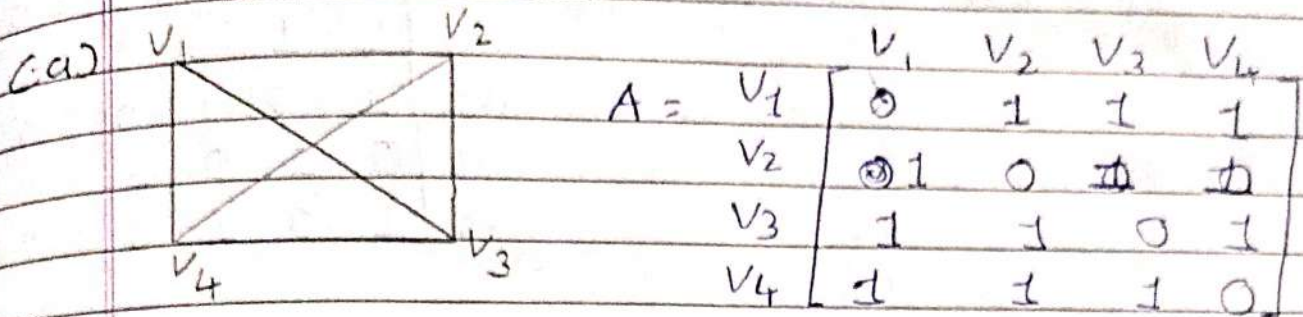
$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 2 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 2 \end{bmatrix}$$

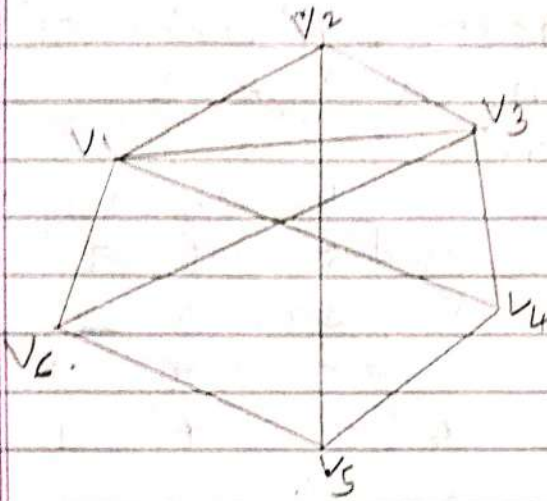
For total number of paths calculate upper triangle of matrix.

$$\text{total path} = 16$$

3 Construct Adjacency Matrix of all graphs shown in Task: 2

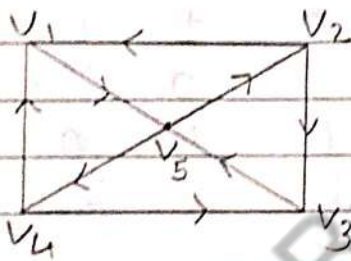


(d)



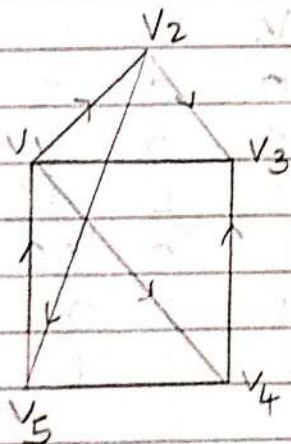
$$A = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

(e)



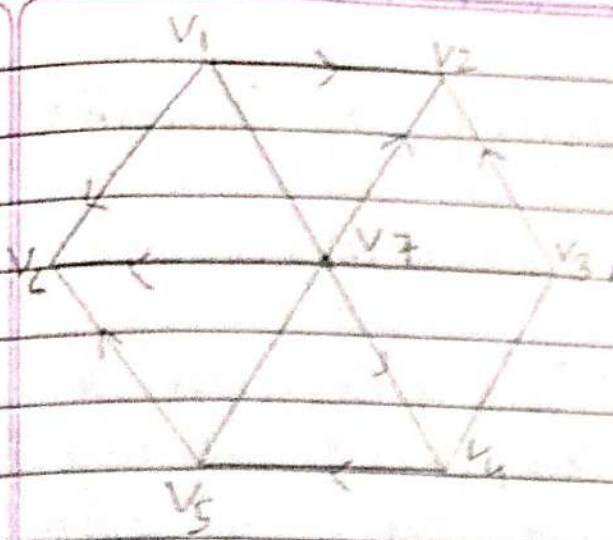
$$A = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 & V_5 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

(f)



$$A = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 & V_5 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

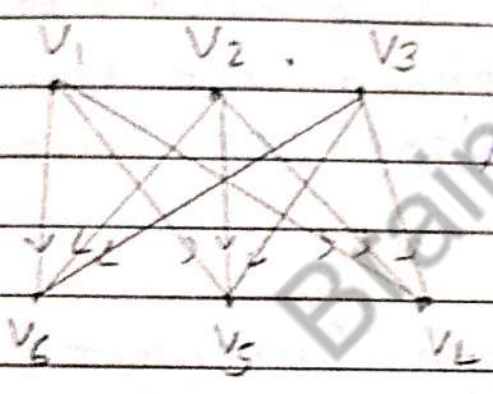
(g)



	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	0	1	0	0	0	0	0
V_2	0	0	0	0	0	0	0
V_3	0	1	0	0	0	0	0
V_4	0	0	0	0	1	0	0
V_5	0	0	0	0	0	1	0
V_6	0	0	0	0	0	0	0
V_7	0	0	0	1	0	1	0

$A =$

(h)



	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	0	0	1	1	1
V_2	0	0	0	1	1	1
V_3	0	0	0	1	1	1
V_4	0	0	0	0	0	0
V_5	0	0	0	0	0	0
V_6	0	0	0	0	0	0

$A =$

* Task : 6 : Coloring of Graphs :

1 Define Following terms:

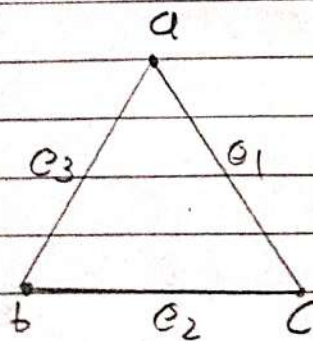
(i) Vertex Coloring: In a Graph G , there are no adjacent vertex color with same, this is called vertex coloring.

(ii) Edge Coloring: In a edge coloring, in Graph G , there are two adjacent vertex does not have same color.

(iii) Chromatic Number: Chromatic number is a minimum number of color that required to color of graph.

2 Coloring Following Graphs:

(a)



For vertex, $a = \text{red}$

$b = \text{green}$

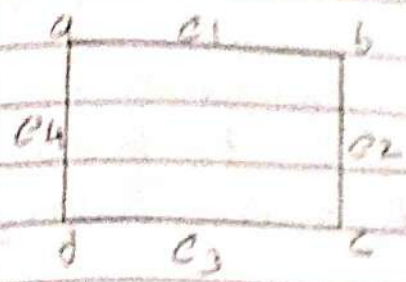
$c = \text{yellow}$

For edges $e_1 = \text{Blue}$

$e_2 = \text{Pink}$

$e_3 = \text{Orange}$

(b)

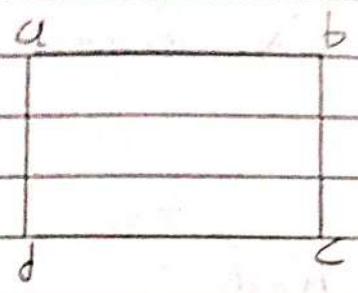


For Vertex - a, c - Red
b, d - Green

For edges - e1, e3 - Yellow
e4, e2 - Orange

3 Find the Chromatic number in the Graph.

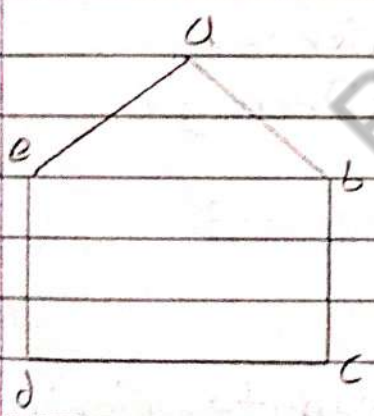
(i)



For Vertex - a, c - Red
b, d - Green

Chromatic number = 2

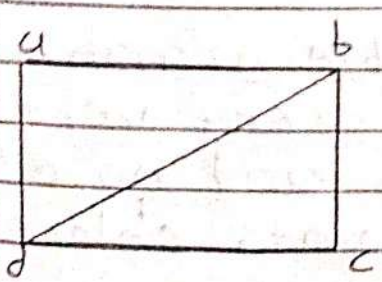
(ii)



For Vertex - a, c - Red
e, b - Green
d - Orange

Chromatic number = 3

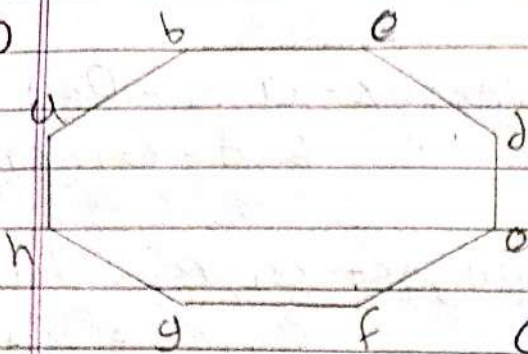
(iv)



For Vertex - a, c - Red
b - Green
d - Orange

Chromatic number = 3

ciii)



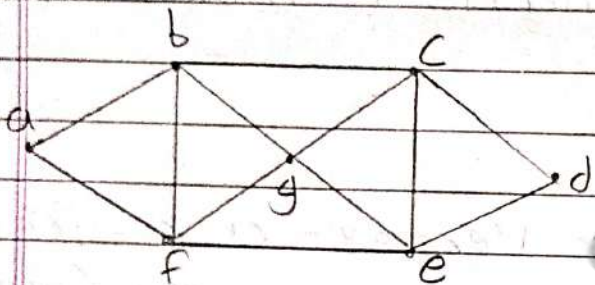
For Vertex:

a, e, c, g - Red

b, d, f, h - Orange

Chromatic Number = 2

civ)



For Vertex:

~~a, c - Red~~~~f, d - Green~~

For Vertex: g, a, d - Red

e, b - Green

c, f - Orange

Chromatic Number = 3

4 Prove that every Bipartite Graph is 2 colorable:

Let G be a 2 colorable graph, which means we can color every vertex into 2 color exactly and no edges will have both and points colored the same color.

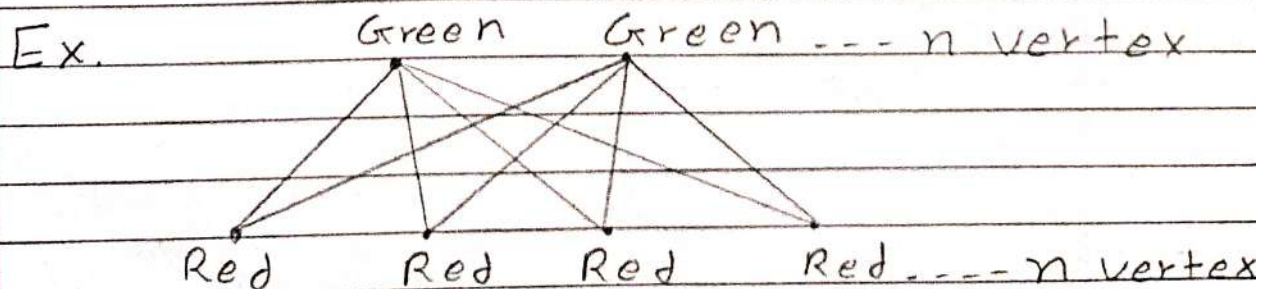
Let a denote the subset of vertices coloured green and Let b denote the subset of vertices coloured Red.

Since all vertices a are green, there are no edges within a and same for b . This implies that every edge has one end point in a and other in b which means G is bipartite.

Conversely, Suppose G is Bipartite, we can partition into two subsets V_1, V_2 . So each edge has one end point in V_1 and the other in V_2 .

Then coloring every vertex of V_1 as Green and that V_2 as blue colour.

Hence, every Bipartite Graph is 2-colorable.



* Task : 3 : Walk, Path and Circuits of Graphs and Diagraphs.

1 Define following terms for Undirected Graph.

(a) Walk: Walk is a finite sequence of vertices and edges beginning and ending with vertex.

(b) Path: In a walk not vertex and not edges are repeated, may be starting vertex are repeated it is called Path.

(c) Circuit: In a trail beginning and ending same this is called circuit.

(d) Length of Path: Number of edges appear in the path, this is called Length of Path.

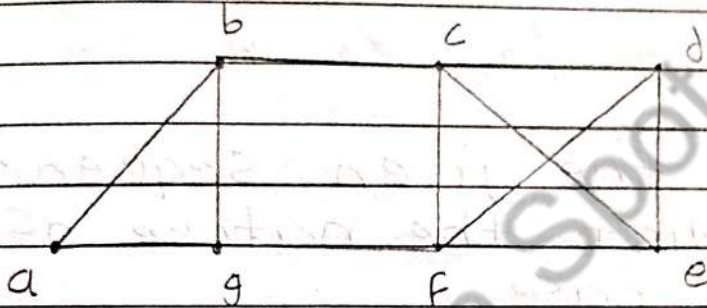
(e) Trail: In a walk edges are not repeated there we can call it is trail.

(f) Euler Graph: If connected graph G is called euler graph, if there

is a closed path trial which includes every edges of the graph.

(g) Hamiltonian Path: A Hamiltonian path is a path that visits each vertex of the graph only one time.

2 Consider the following Path.



Decide which of the following is walks, circuit, a path, a cycle or a trail.

(a) a, b, g, f, c, b - Trail

(b) b, g, f, c, b, g, a - Walk

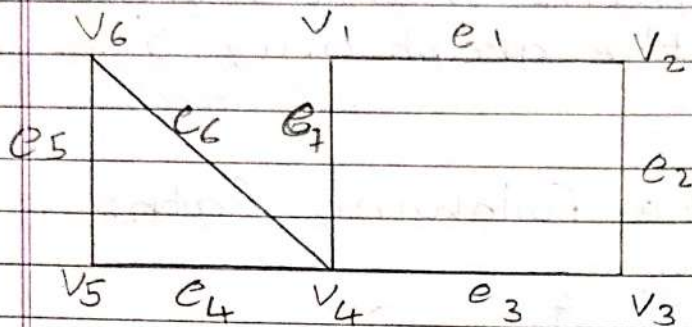
(c) c, e, f, c - Cycle

(d) c, e, f, c, e - Walk

(e) a, b, f, a - Not a walk

(F) f, d, e, c, b - Path

3 Consider the following graph:



Observe the given sequences and predict the nature of walk in each case.

(a) $V_1 e_1 V_2 e_2 V_3 e_2 V_2$ - Open Walk

(b) $V_4 e_7 V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_4 V_5$ - Trail

(c) $V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_4 V_5$ - Path

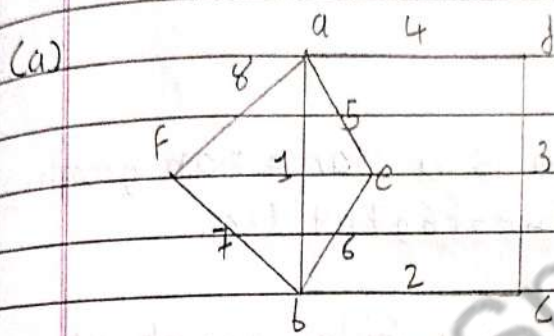
(d) $V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_7 V_1$ - Cycle

(e) $V_6 e_5 V_5 e_4 V_4 e_3 V_3 e_2 V_2 e_1 V_1 e_7 V_4 e_6 V_6$

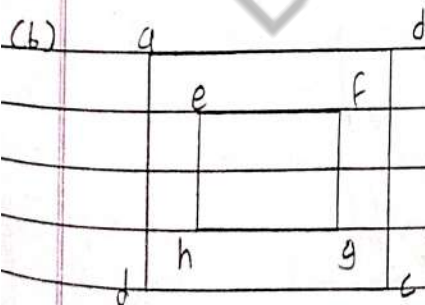
Path Circuit

4 Which of the following graphs are Eulerian? If they are Eulerian, mention Euler circuit.

Also which of them is Hermitian? If they are Hermitian, mention hermitian circuit.

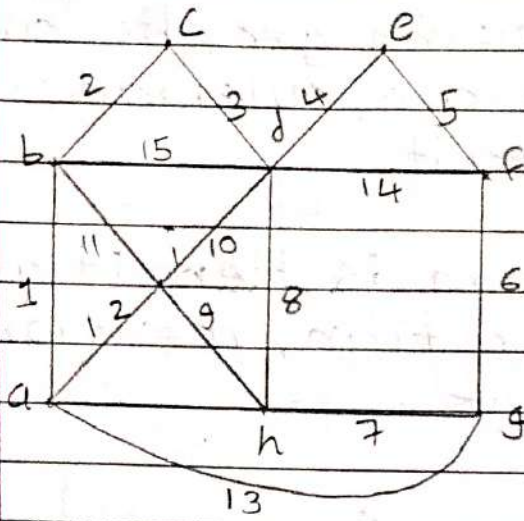


This Graph is Euler Graph as
762c3d4a5e6b7f



Here, This graph is not Euler or Hermitian graph.

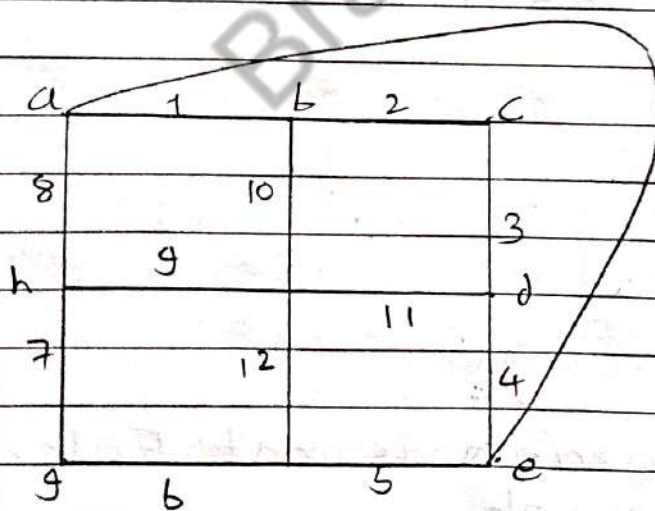
Cc)



This graph is a Hamiltonian graph as $1b2c3d4e5f6g7h9i12a$.

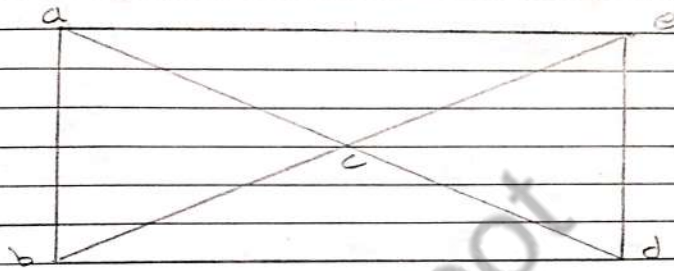
This graph is a Euler graph as $f14d15b2c3de5f6g7hrd10i12a1611i9h16a13g$.

Cd)

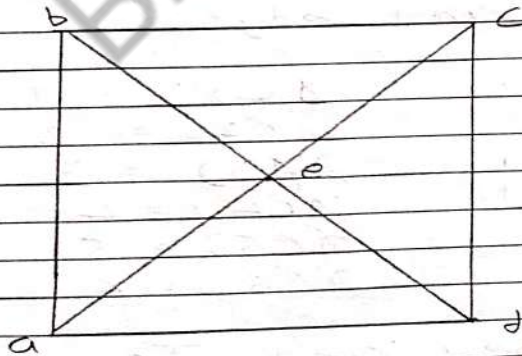


This graph is Hamiltonian graph as $1b2c3d11i9h7g6f5e$.

5 Give an example of a graph which is Eulerian but not Hamiltonian.

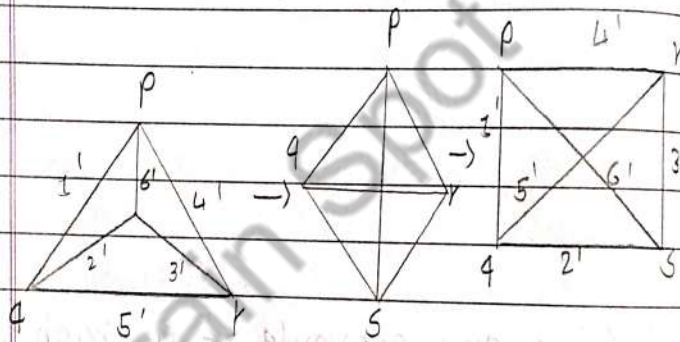
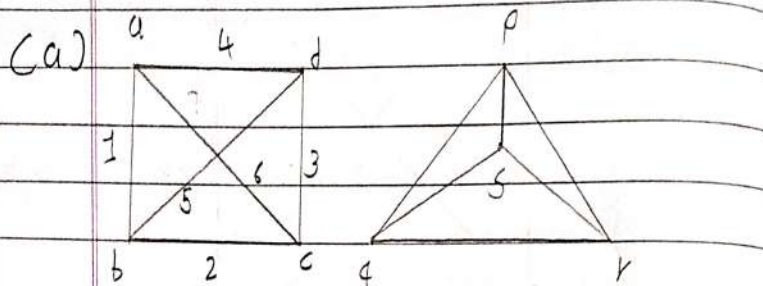


6 Give one example of a graph which is Hamiltonian but not Eulerian.



* Task: 2: Isomorphism of Graphs and Diagraphs.

1 Prove that following pair of graphs are isomorphic.

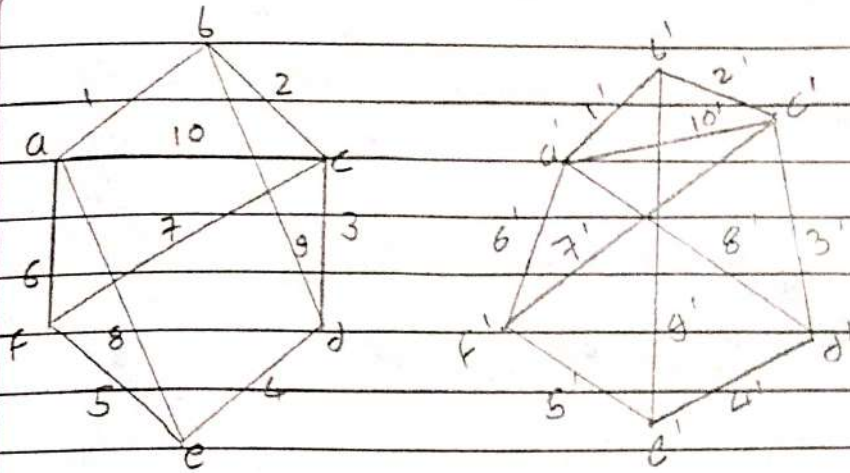


They have same number of vertex and edges.

- $\phi(p) = a$ $\phi'(1') = 1$
- $\phi(q) = b$ $\phi'(2') = 2$
- $\phi(r) = d$ $\phi'(3') = 3$
- $\phi(s) = c$ $\phi'(4') = 4$
- $\phi'(5) = 5$
- $\phi'(6') = 6$

Thus they are Isomorphic.

(b)



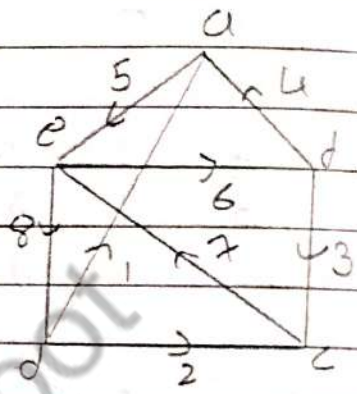
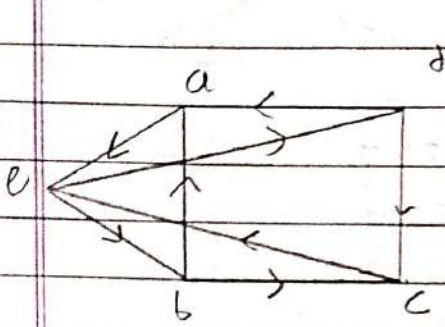
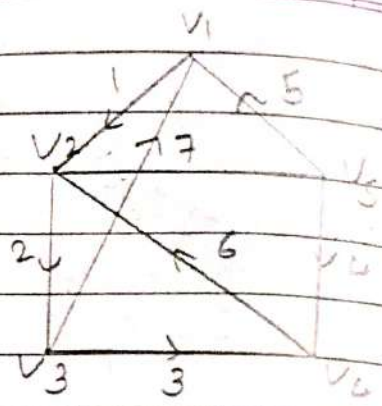
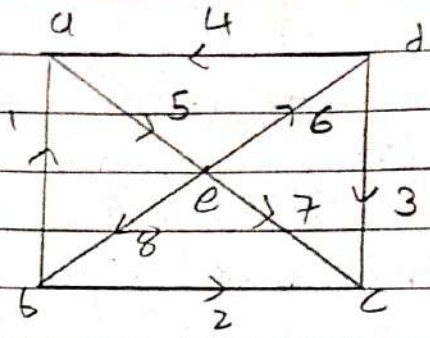
$$\begin{aligned} \rightarrow \partial(a) &= 4 & \partial(a') &= 4 \\ \partial(b) &= 3 & \partial(b') &= 3 \\ \partial(c) &= 3 & \partial(c') &= 3 \\ \partial(d) &= 3 & \partial(d') &= 3 \\ \partial(e) &= 4 & \partial(e') &= 4 \\ \partial(f) &= 3 & \partial(f') &= 3 \end{aligned}$$

Both graph has same number of vertex and edges.

$$\begin{aligned} \phi(a) &= a' & \phi(1) &= 1' \\ \phi(b) &= b' & \phi(2) &= 2' \\ \phi(c) &= c' & \phi(3) &= 3' \\ \phi(d) &= d' & \phi(4) &= 4' \\ \phi(e) &= e' & \phi(5) &= 5' \\ \phi(f) &= f' & \phi(6) &= 6' \end{aligned}$$

Hence, both graphs are isomorphic.

(c)



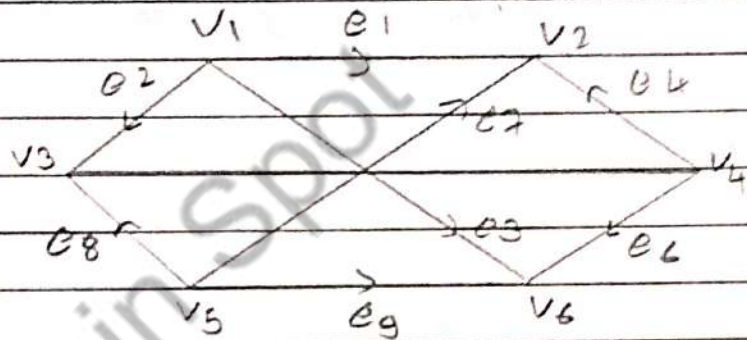
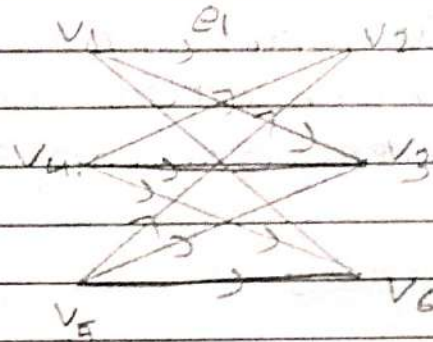
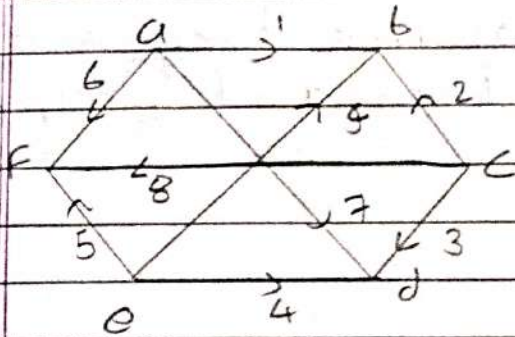
- | | |
|-------------------|---------------------|
| $\partial(a) = 1$ | $\partial(V_1) = 1$ |
| $\partial(b) = 2$ | $\partial(V_2) = 2$ |
| $\partial(c) = 1$ | $\partial(V_3) = 1$ |
| $\partial(d) = 2$ | $\partial(V_4) = 2$ |
| $\partial(e) = 2$ | $\partial(V_5) = 2$ |

Both Graph have same number of vertex and edges.

- | | |
|-----------------|----------------|
| $\phi(a) = V_1$ | $\phi'(1) = 5$ |
| $\phi(b) = V_3$ | $\phi'(2) = 8$ |
| $\phi(c) = V_4$ | $\phi'(3) = 2$ |
| $\phi(d) = V_5$ | $\phi'(4) = 4$ |
| $\phi(e) = V_2$ | $\phi'(5) = 4$ |
| | $\phi'(6) = 2$ |
| | $\phi'(7) = 1$ |
| | $\phi'(8) = 6$ |

Hence, Both Graphs are isomorphic.

(d)



There are same number of vertex and edges in graph.

$\phi(a) = v_1$	$\phi'(1) = e_1$
$\phi(b) = v_2$	$\phi'(2) = e_4$
$\phi(c) = v_4$	$\phi'(3) = e_6$
$\phi(d) = v_6$	$\phi'(4) = e_9$
$\phi(e) = v_5$	$\phi'(5) = e_8$
$\phi(f) = v_3$	$\phi'(6) = e_2$
	$\phi'(7) = e_3$
	$\phi'(8) = e_5$
	$\phi'(9) = e_7$

Hence, both Graphs are isomorphic.

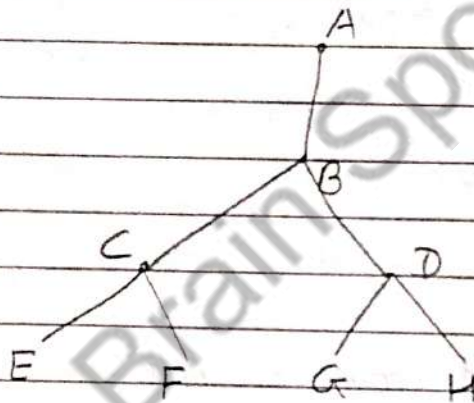
* Task - 7 : Tree.

1 Define the following terms of undirected and directed graph with example.

(a) Tree:

A connected graph that contains no cycle is called tree.

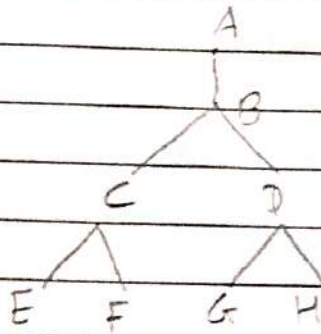
Ex.



(b) Binary Tree:

A binary tree is a tree-like structure that is rooted and in which each vertex has at most two children.

Ex



(c) Terminal vertex:

In a directed tree any vertex which has out degree zero is called a terminal vertex.

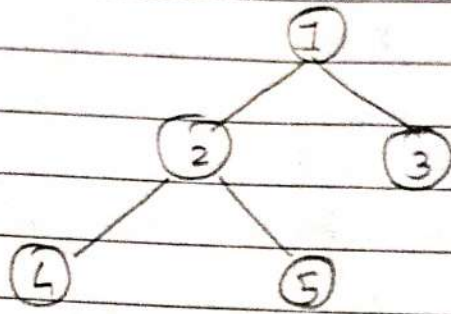
- Internal vertex:

In a tree, vertex that have children are called internal vertex.

(d) height of tree:

The height of a tree is defined as the maximum number of nodes in a branch of a tree.

Ex.



Height of tree = 3

Brain Spot