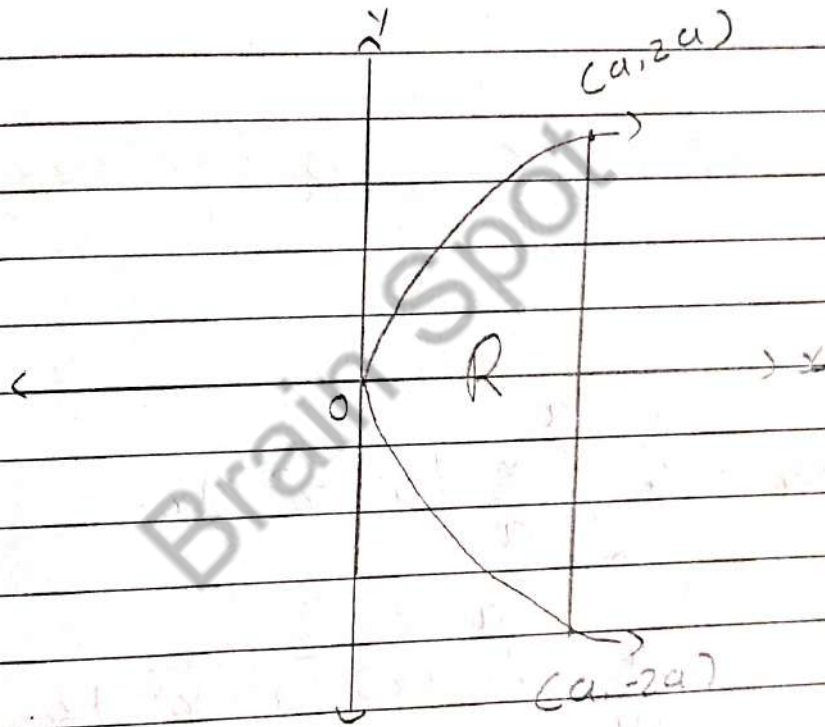


## Unit-5 Integral Calculus

### \* Task 1 : Area

1 Find the area between the parabola  $y^2 = 4ax$  and its latus rectum.



$$\text{Area} = 2 \int_0^a \sqrt{2\sqrt{ax}} \cdot dx$$

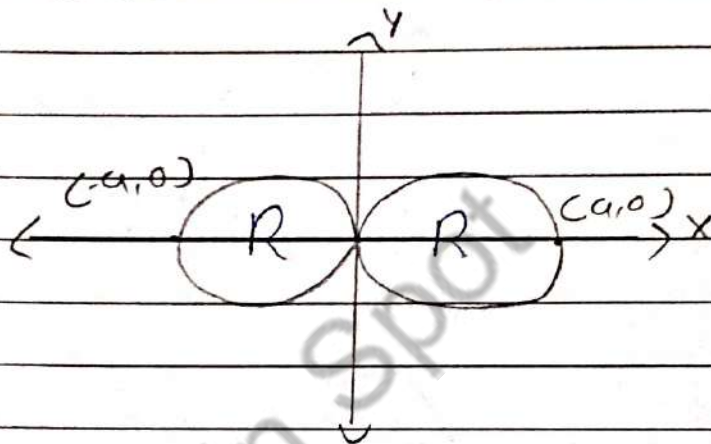
$$= 2 \int_0^a \sqrt{2a} \cdot x^{1/2} dx$$

$$= 4\sqrt{a} \left[ \frac{x^{3/2}}{3/2} \right]$$

$$\text{Area} = \frac{8}{3} \sqrt{a} \cdot a^{3/2}$$

$$A = \frac{8}{3} a^3$$

2 Find the area of the loops of the curve  $a^2 y^2 = x^2 (a^2 - x^2)$



$$\text{Area} = 4 \int_0^a \frac{x}{a} \sqrt{a^2 - x^2} dx$$

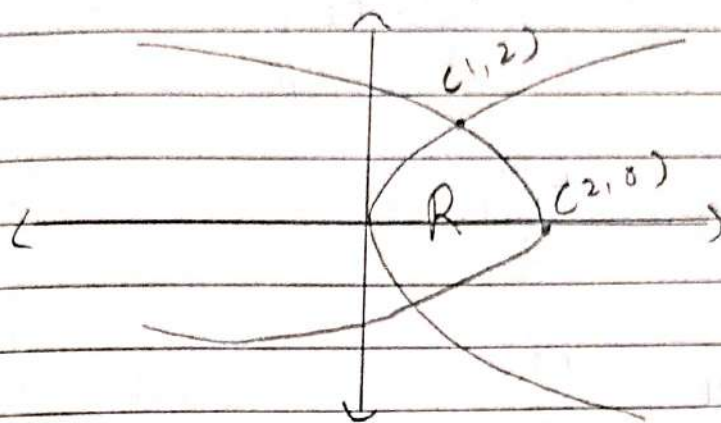
$$= \frac{-4}{2a} \int_0^a (-2x) \sqrt{a^2 - x^2} dx$$

$$= \frac{-2}{a} \left[ \frac{2(a^2 - x^2)^{3/2}}{3} \right]_0^a$$

$$= \frac{-2}{a} \left[ \frac{2(-a^2)^{3/2}}{3} \right]$$

$$\text{Area} = \frac{4a^3}{3}$$

- 3 Find the area enclosed by  $y^2 = 4x$  and  $y^2 = -4(x-2)$



$$x = \frac{y^2}{4} \quad \text{and} \quad x = \frac{8-y^2}{4}$$

$$\text{Area} = 2 \int_0^2 \int_{\frac{y^2}{4}}^{\frac{8-y^2}{4}} 1 \, dx \, dy$$

$$= 2 \int_0^2 [x]_{\frac{y^2}{4}}^{\frac{8-y^2}{4}} dy$$

$$= 2 \int_0^2 \frac{y^2}{4} - \frac{8-y^2}{4} dy$$

$$= \frac{1}{2} \int_0^2 2y^2 - 8 \, dy$$

$$= \frac{1}{2} \left[ \frac{2y^3}{3} - 8y \right]_0^2$$

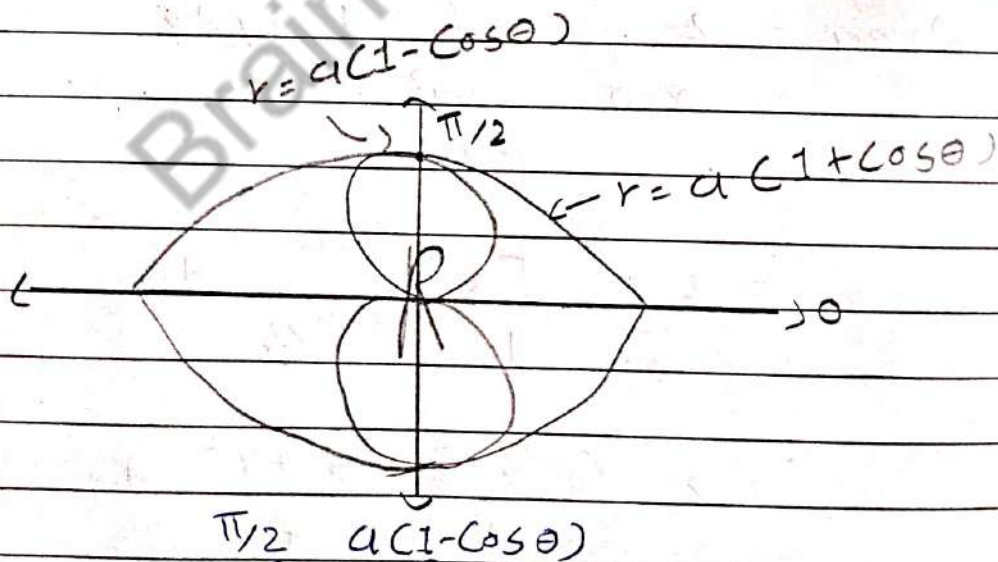
$$\text{Area} = \frac{1}{2} \left[ \frac{2(8) - 8(2)}{3} \right]$$

$$= \frac{1}{2} \left[ \frac{16 - 16}{3} \right]$$

$$= \frac{+32}{6}$$

$$\text{Area} = \frac{16}{3}$$

4 Find the area between the the cardioids  $r = a(1 + \cos\theta)$  and  $r = a(1 - \cos\theta)$



$$\text{Area} = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{a(1-\cos\theta)} r \, dr \, d\theta$$

$$= 4 \int_0^{\pi/2} \left[ \frac{r^2}{2} \right]_0^{a(1-\cos\theta)} d\theta$$

$$= 4 \int_0^{\pi/2} \frac{1}{2} a^2 (1 - \cos\theta)^2 d\theta$$

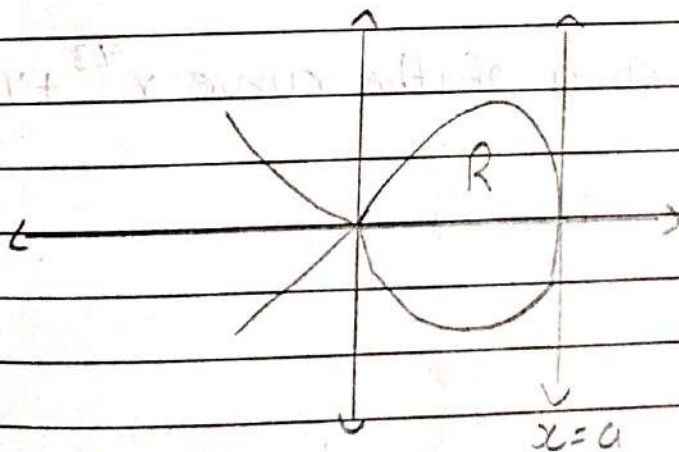
$$= 2a^2 \int_0^{\pi/2} 1 - 2\cos\theta + \cos^2\theta d\theta$$

$$= 2a^2 \left[ \theta - 2\sin\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{2a^2}{2} \left[ \frac{3\pi}{4} - 2 \right]$$

$$\text{Area} = a^2 \left[ \frac{3\pi}{2} - 4 \right]$$

5 Find the area of the loop of  $ay^2 = x^2(a-x)$  in first quadrant only.



$$\therefore y = x \sqrt{\frac{a-x}{a}}$$

$$A = \int_0^a x \sqrt{\frac{a-x}{a}} dx$$

$$A = \frac{1}{\sqrt{a}} \int_0^a (a-x) \sqrt{a-x} dx$$

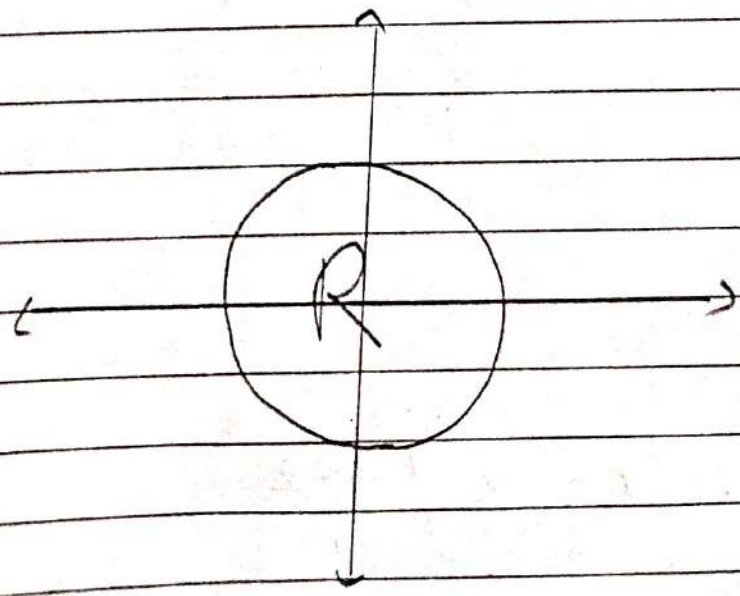
$$A = \frac{1}{\sqrt{a}} \int_0^a a \sqrt{a-x} - (a-x)^{3/2} dx$$

$$A = \frac{1}{\sqrt{a}} \left[ \frac{-2a}{3} (a-x)^{3/2} + \frac{2}{5} (a-x)^{5/2} \right]_0^a$$

$$= \frac{1}{\sqrt{a}} \left[ \frac{2}{3} a^{5/2} - \frac{2}{5} a^{5/2} \right]$$

$$= \frac{a^2 \cdot 4}{15}$$

6 Find the area of the curve  $x^{2/3} + y^{2/3} = a$



$$\text{Here, } x = a \cos^3 \theta, \quad y = a \sin^3 \theta$$

$$\therefore dx = -3a \cos^2 \theta \sin \theta$$

$$A = 4 \int_0^{\pi/2} y \, dx$$

$$= 4 \int_0^{\pi/2} a \sin^3 \theta (-3a \cos^2 \theta \sin \theta) \, d\theta$$

$$= 4(-3a^2) \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta \, d\theta$$

$$A = -12a^2 \frac{\beta}{2} \left( \frac{5}{2}, \frac{3}{2} \right)$$

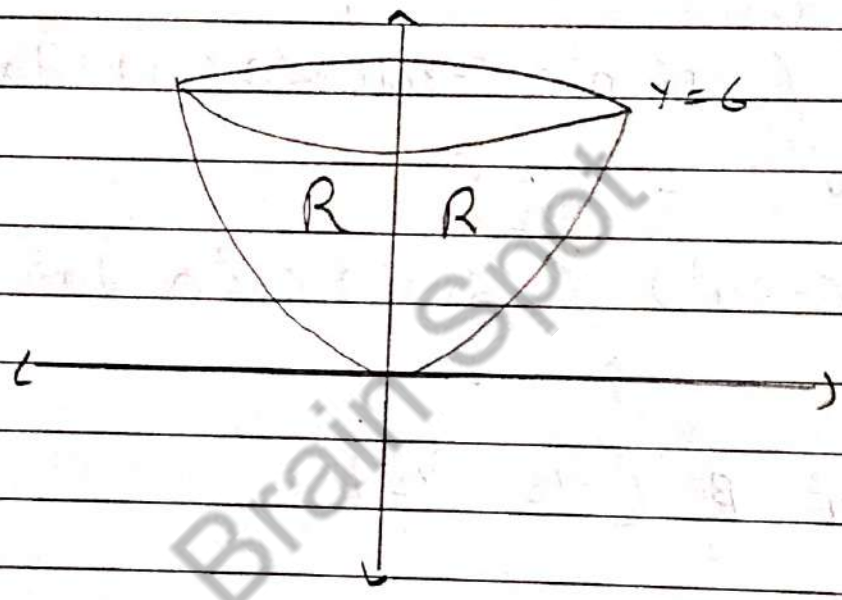
$$A = -6a^2 \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{2} \times \sqrt{\pi}$$

3 × 2 × 1

$$A = \frac{3\pi a^2}{8}$$

\* Task-2 : Volume of solid of Revolution

1 Find the volume of the solid obtained by rotating the region bounded  $y = x^2$ ,  $y = 6$  and  $x = 0$  about  $y$ -axis.



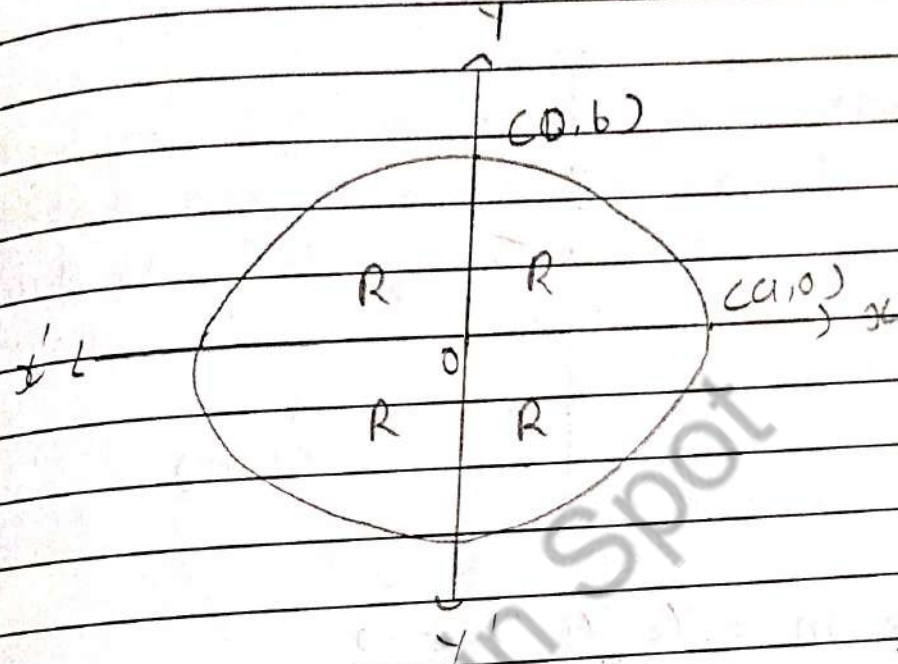
$$\begin{aligned}
 \text{Volume} &= \int_0^6 \pi x^2 dy \\
 &= \int_0^6 y^2 \pi dy \\
 &= \pi \left[ \frac{y^3}{3} \right]_0^6
 \end{aligned}$$

$V = 18\pi$



2 Find the volume of the solid obtained by rotating the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  about

$x$ -axis.



$$V = 2 \int_0^a \pi y^2 dx$$

$$= 2\pi \int_0^a \left( b^2 - \frac{b^2 x^2}{a^2} \right) dx$$

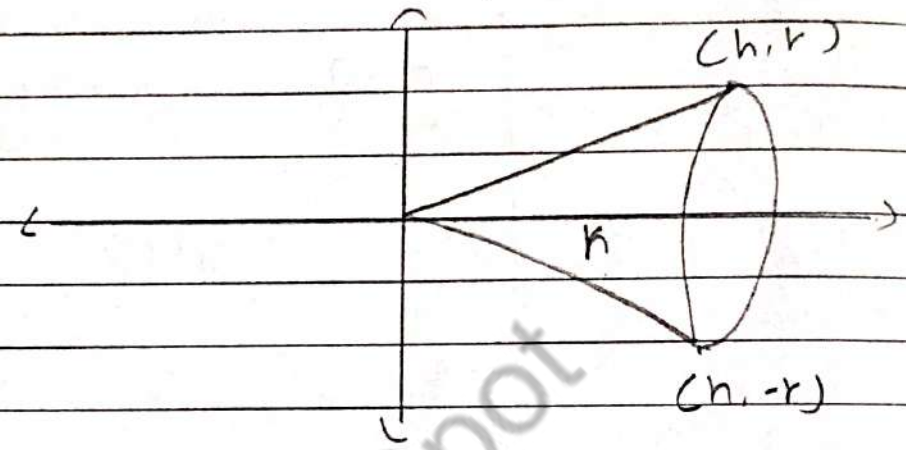
$$= 2\pi b^2 \int_0^a \left( 1 - \frac{x^2}{a^2} \right) dx$$

$$= 2\pi b^2 \left[ x - \frac{x^3}{3a^2} \right]_0^a$$

$$= 2\pi b^2 \left[ a - \frac{a^3}{3a^2} \right]$$

$$V = \frac{4b^2\pi a}{3}$$

3 Find the volume of cone of height  $h$  and base  $r$ .



$$\Rightarrow \text{Slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{r - 0}{h - 0}$$

$$\therefore x_2 - x_1 = \frac{h}{r} (y_2 - y_1)$$

$$\therefore \frac{x_2}{h} = y$$

$$\text{Volume} = \int_0^h \pi y^2 dx$$

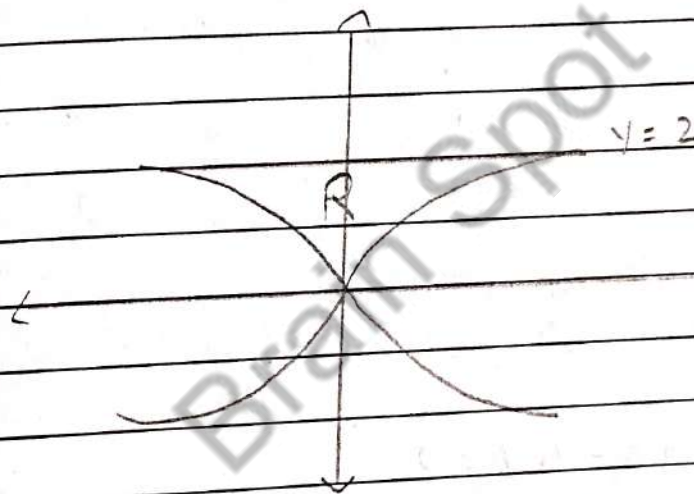
$$= \pi \int_0^h \left( \frac{xr}{h} \right)^2 dx$$

$$= \frac{\pi r^2}{h^2} \left[ \frac{x^3}{3} \right]_0^h$$

$$V = \frac{r^2 \pi}{h} \left[ \frac{h^3}{3} \right]$$

$$V = \frac{\pi r^2 h}{3}$$

4 Find the volume of the solid generated by rotating the region bounded by  $y = \sqrt{x}$  and the line  $y=2$ ,  $x=0$  about the line  $x=0$ .



Here  $y = \sqrt{x}$

$$\therefore x^2 = y^4$$

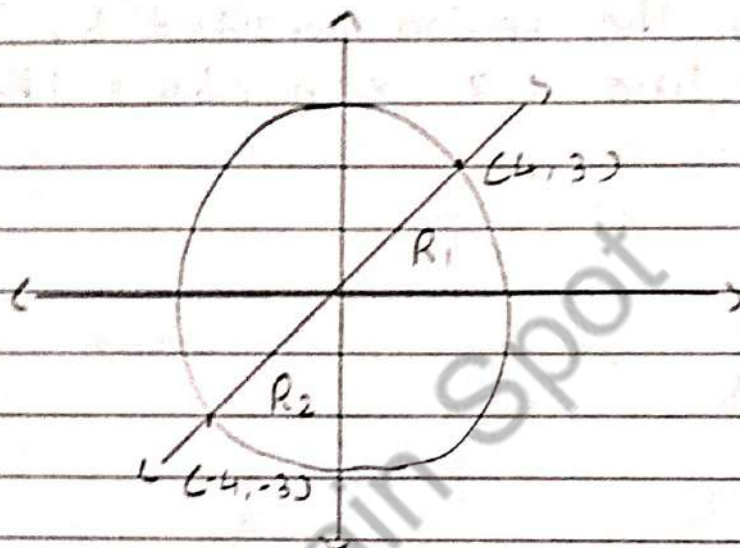
$$\text{Volume} = \int_0^2 \pi x^2 dy$$

$$= \pi \int_0^2 y^4 dy$$

$$= \pi \left[ \frac{y^5}{5} \right]_0^2$$

$$V = \frac{32\pi}{5}$$

5 Find the volume generated by rotating about  $x$ -axis the area bounded by the curve  $x^2 + y^2 = 25$ ,  $3x - 4y = 0$  and  $y = 0$ .



$$\Rightarrow \text{Here } 3x - 4y = 0$$

$$x = \frac{4y}{3}$$

$$\therefore x^2 + \left[ \frac{3x}{4} \right]^2 = 25$$

$$\Rightarrow 3x - 4y = 0 \quad \therefore y = \frac{3x}{4}$$

$$\therefore x^2 + \left( \frac{3x}{4} \right)^2 = 25$$

$$\therefore x = \pm 4$$

$$\Rightarrow x = +4 \quad \text{For } y = \frac{3(4)}{4} = 3$$

$$x = -4 \quad \text{For } y = \frac{3(-4)}{4} = -3$$

$$\Rightarrow \text{Volume} = 2 \left[ \int_0^4 y_1^2 \pi dx + \int_4^5 y_2^2 \pi dx \right]$$

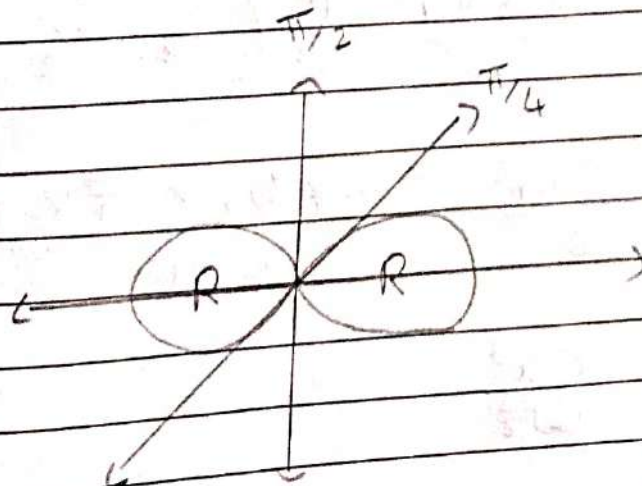
$$= 2 \left[ \int_0^4 \pi \left( \frac{3x}{4} \right)^2 dx + \int_4^5 \pi (25 - x^2) dx \right]$$

$$= 2\pi \left( \frac{9}{16} \left[ \frac{x^3}{3} \right]_0^4 + \left[ 25x - \frac{x^3}{3} \right]_4^5 \right)$$

$$= 2\pi \left( \frac{9}{16} \left[ \frac{64}{3} \right] + \left( 25(5) - \frac{5^3}{3} \right) - \left( 25(4) - \frac{4^3}{3} \right) \right)$$

$$= \frac{100\pi}{3}$$

6 Find the volume of the solid generated by revolving the curve  $r^2 = a^2 \cos 2\theta$



$$\text{Volume} = 2 \int_0^{\pi/4} \frac{2\pi}{3} r^3 d\theta \cdot \cos\theta$$

$$= \frac{4\pi}{3} \int_0^{\pi/4} a^3 (\cos 2\theta)^{3/2} \cos\theta d\theta$$

$$= \frac{4\pi}{3} a^3 \int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta d\theta$$

Suppose  $\sqrt{2}\sin\theta = \sin\phi$

$\therefore \sqrt{2}\cos\theta d\theta = \cos\phi d\phi$

$\theta \rightarrow \frac{\pi}{4} \rightarrow \phi = \frac{\pi}{2}$

$\theta \rightarrow 0 \rightarrow \phi = 0$

$$\text{Volume} = \frac{4\pi a^3}{3} \int_0^{\pi/2} (1 - \sin^2\phi)^{3/2} \cdot \frac{\cos\phi d\phi}{\sqrt{2}}$$

$$= \frac{2\sqrt{2}\pi a^3}{3} \int_0^{\pi/2} \cos^3\phi \cdot \sin\phi d\phi$$

$$= \frac{2\sqrt{2}\pi a^3}{3} \frac{\beta}{2} \left( \frac{1}{2}, \frac{5}{2} \right)$$

$$V = \frac{2\sqrt{2}\pi a^2}{6} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} \times \sqrt{\pi}$$

$$V = \frac{\pi^2 a^3}{4\sqrt{2}}$$