Unit - 2 : Lattices

* Task : 1

1 Define Poset with Example.

Let a relation define on non-empty set x is partially Ordered r^n then, this set is called "Partially Order Set".

Poset are denoted by $\langle x, \leqslant \rangle$

Ex. Suppose A is any non-empty set and show that $\langle P(A), \leqslant \rangle$ is partially Ordered r^n.

-> Here, given set is all the subset of A which is $P(A)$ and r^n, "$\leqslant$".

(a) Reflexive:

$\forall A_1 \in P(A)$, We write $A_1 \leqslant A_1$

So, "$\leqslant$" is Reflexive.

(b) Anti symmetric:

$\forall A_1, A_2 \in P(A)$,

Suppose $A_1 \leq A_2$ and $A_2 \leq A_1$
then always $A_1 = A_2$.

" $\leq$ " is antisymmetric.

(c) Transitive:

$\forall A_1, A_2, A_3 \in P(A)$,

If $A_1 \leq A_2$, $A_2 \leq A_3$ then
$A_1 \leq A_2$ is possible.

" $\leq$ " is transitive.

Relation " $\leq$ " is Paratially Ordered
set on R^n.

Hence, $\langle P(A), \leq \rangle$ is Poset.

2 Consider the set Z of integer. Define aRb is there is a positive integer r such that $b = a^r$, is R a Poset on Z?

→ Here, Given set $= Z$ and
aRb if $b = a^r$, $r \in Z^+$

For Poset on Z must be satisfy this three properties.

(1) Reflexive: $\forall, a \in Z$

if aRa then $a = a^r$ for $r = 1 \in Z^+$
$\therefore$ R is Reflexive.

(2) Anti-symmetric: $\forall a, b \in Z$

if aRb then $a = b^r$ or $b = a^r$
for $r = 1 \in Z^+$

$\therefore$ R is anti-symmetric.

(3) Transitive: $\forall a, b, c \in Z$

if aRb and bRc then,

$\cancel{a =} b = a^r$, $c = b^r$

$\therefore$ $c = (a^r)^r$ for $r = 1 \in Z^+$
$\therefore$ aRc

$\therefore$ R is transitive.

Hence, $b = r^n$ is a Poset of Z.

3 Let $N = \{1, 2, 3, \ldots\}$ be ordered by D. State whether each of the following subset of N are lineary Ordered.

(a) $\{24, 2, 6\}$

$R = \{(24, 2), (2, 24), (24, 6), (6, 24), \cancel{(2, 24)}, \cancel{(2, 6)}, (6, 2), \}$

Here, every Pair is follows $\frac{a}{b}$ or $b/a \in N$ set.

$\therefore$ Subset of N are totally Ordered.

(b) $\{3, 15, 5\}$

$R = \{(3, 15), (3, 5), (15, 5), (15, 3), (5, 3), (5, 15)\}$

Here, every Pair is not follows a/b or $b/a \notin N$ set.

So, Subset of N are not totally Ordered.

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(d) $\{2, 8, 32, 4\}$

$R = \{(2,8), (2,32), (2,4), (8,32), (8,4), (8,2), (32,4), (32,2), (32,8), (4,2), (4,8), (4,32)\}$

Here, every Pair is not follows a/b or $b/a \in N$.

So, ~~N~~ Subset of N are not lineary Ordered.

(c) $\{1, 2, 3, ---\}$

$R = \{(1,2), (1,3), (1,4), ---\}$

Here, every Pair is not follows a/b or $b/a \in N$.

So, Subset of N are not lineary Ordered.

(e) $\{7\}$

Here, Given set is not containt other element.

So, Subset of N are lineary Ordered.

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(f) $\{15, 5, 30\}$

$R = \{(15,5), (15,30), (5,30), (5,15), (30,15), (30,5)\}$

Here, every pair follows a/b or $b/a \in N$.

So, Subset of N is linearly Ordered.

4 Show that $\langle P(x), \subseteq \rangle$ is a poset where X is non empty set and P(x) is a poset set of X.

Here Given set is all the subset of x.

For Poset, on X must be follow this three condition.

(a) Reflexive: For $\forall x \subseteq P(x)$,

$\therefore x_1 \subseteq x_1$

$\therefore$ "$\subseteq$" is Reflexive.

(b) Antisymmetric: For $\forall x_1, x_2 \in P(x)$.

if $x_1 \subseteq x_2$ then $x_2 \subseteq x_1$ and always $x_1 = x_2$.

"$\subseteq$" is antisymmetric.

(c) Transitive: $\forall x_1, x_2, x_3 \in P(A)$

if $x_1 \subseteq x_2$, and $x_2 \subseteq x_3$ then for subset of power set is possible $x_1 \subseteq x_3$.

"$\subseteq$" is Transitive.

Hence, $\langle P(x), \subseteq \rangle$ is Poset.

5 Which of the following are Posets?

(a) $\langle Z, = \rangle$

For Poset,

(i) Reflexive: $\forall a \in Z$, $a = a$

$\therefore$ "=" is Reflexive.

(ii) Antisymmetric: $\forall a, b \in Z$,

$\therefore$ $a = b$ then $b = a$.

$\therefore$ "=" is Antisymmetric.

(iii) Transtivie: $\forall a, b, c \in Z$

if $a = b$ and $b = c$ then always $a = c$

$\therefore$ "$=$" is Transtivie.

Hence, $\langle Z, = \rangle$ is Poset.

(b) $\langle Z, \neq \rangle$

For Poset,

(i) Reflexive: $\forall a \in Z$, always $a = a$.

We can not do $a \neq a$.

So, $\langle Z, \neq \rangle$ is not Poset.

(c) $\langle Z, > \rangle$

For Poset,

Reflexive: $\forall a \in Z$, always $a = a$, there is not be $a > a$.

So, $\langle Z, > \rangle$ is not Poset.

(d) $\langle Z, < \rangle$

For Poset,

Reflexive: $\forall a \in Z$, always $a = a$.
We can not write $a < a$.

So, $\langle Z, < \rangle$ is not Poset.

6 Show that $\langle \{1, 3, 3^2, 3^3 \dots\}, D \rangle$ is a chain.

Here, Given set $R = \{1, 3, 3^2, 3^3, \dots\}$

For Poset, there are three Properties must be follow.

(i) Reflexive: $\forall a \in R$,

$\therefore$ a is always D by a.

$\therefore$ "D" is Reflexive.

(ii) Transtivie : $\forall a, b, c \in R$.

if a^r / b^r and b^r / c^r then,

a^r / c^r is must be D by

So, "D" is Transitive.

(iii) Antisymmetric:

$\forall\ a^2, b^2 \in Z$

Suppose, a^2 / b^2 and $b^2 = a^2$
then $a^2 = b^2 \in \cancel{Z} N$

$\therefore$ "D" is Antisymmetric.

Hence, $\langle \{1, 3, 3^2, \ldots\}, D \rangle$ is a chain.

* Task : 2

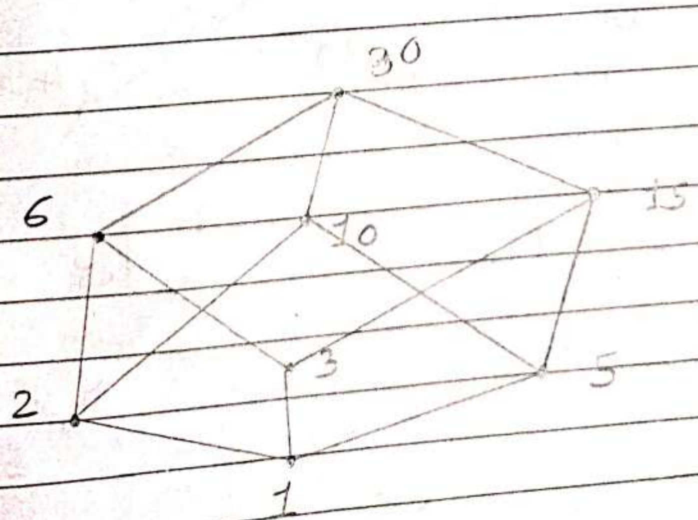
1 Draw the following Hass Diagram.

(a) $\langle S_{30}, D \rangle$

$S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Cover of all the elements,

$1 \rightarrow 2, 3, 5$
$2 \rightarrow 6, 10$
$3 \rightarrow 6, 15$
$5 \rightarrow 10, 15$
$6 \rightarrow 30$
$10 \rightarrow 30$
$15 \rightarrow 30$
$30 \rightarrow \emptyset$

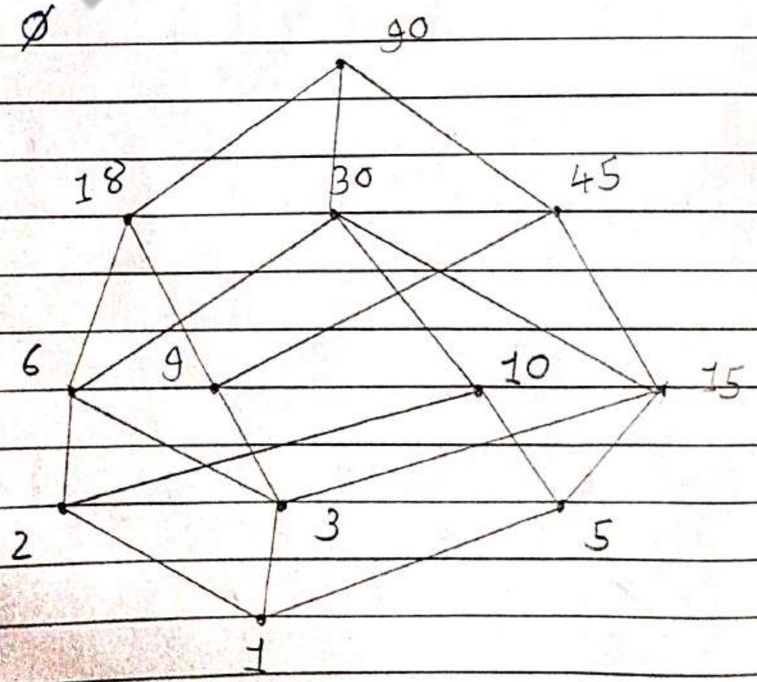


(b) $\langle S_{90}, D \rangle$

$S_{90} = \{1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90\}$

Cover of all the element,

1 -> 2, 3, 5
2 -> 6, 10
3 -> 6, 9, 15
5 -> 10, 15
6 -> 18, 30
9 -> 18, 45
10 -> 30
15 -> 30, 45
18 -> 90
30 -> 90
45 -> 90
90 -> ∅

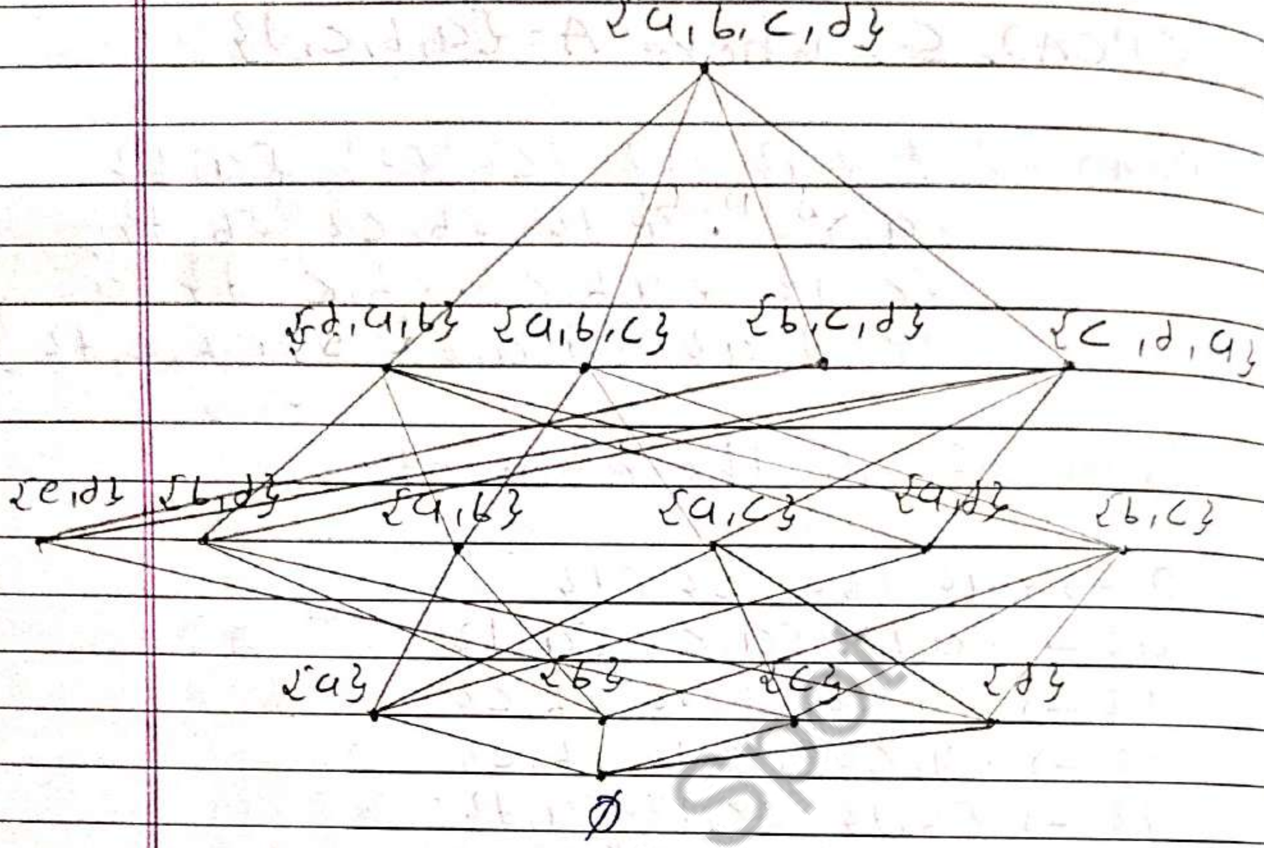


(c) $\langle P(A), \subseteq \rangle$ where $A = \{a, b, c, d\}$

$P(A) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\},$
$\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\},$
$\{c, d\}, \{a, b, c\}, \{b, c, d\},$
$\{c, d, a\}, \{d, a, b\}, \{a, b, c, d\} \}$

Cover of all the element,

$\emptyset \to \{a\}, \{b\}, \{c\}, \{d\}$
$\{a\} \to \{a, b\}, \{a, c\}, \{a, d\}$
$\{b\} \to \{a, b\}, \{b, d\}, \{b, c\}$
$\{c\} \to \{a, c\}, \{c, d\}, \{b, c\}$
$\{d\} \to \{b, d\}, \{c, d\}, \{a, d\}$
$\{a, b\} \to \{a, b, c\}, \{\cancel{b}, d, a\}$
$\{a, c\} \to \{a, b, c\}, \{c, d, a\}$
$\{a, d\} \to \{a, b, d\}, \{a, c, d\}$
$\{b, c\} \to \{a, b, c\}, \{b, c, d\}$
$\{b, d\} \to \{a, b, d\}, \{b, c, d\}$
$\{c, d\} \to \{c, d, a\}, \{b, c, d\}$
$\{a, b, c\} \to A$
$\{b, c, d\} \to A$
$\{c, d, a\} \to A$
$\{d, a, b\} \to A$
$\{a, b, c, d\} \to X$



(d) $\langle S_{210}, D \rangle$

$S_{210} = \{1, 2, 3, 5, 6, 7, 10, 21, 30, 35, 42, 70, 105, 210\}$

Cover of the all the element.

1 → 2, 3, 5, 7
2 → 6, 10
3 → 6, 21
5 → 10, 35
6 → 30, 42
7 → 21, 35
10 → 30, 70

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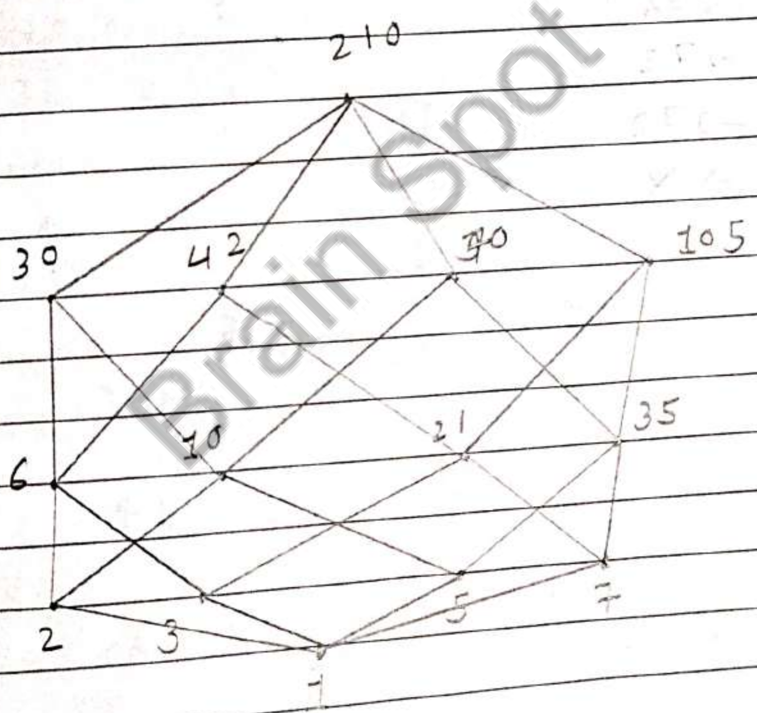
21 → 42, 105
30 → 210
35 → 70, 105
42 → 210
70 → 210
105 → 210
210 → X



(e) $\langle S_{72}, D \rangle$

$S_{72} = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$

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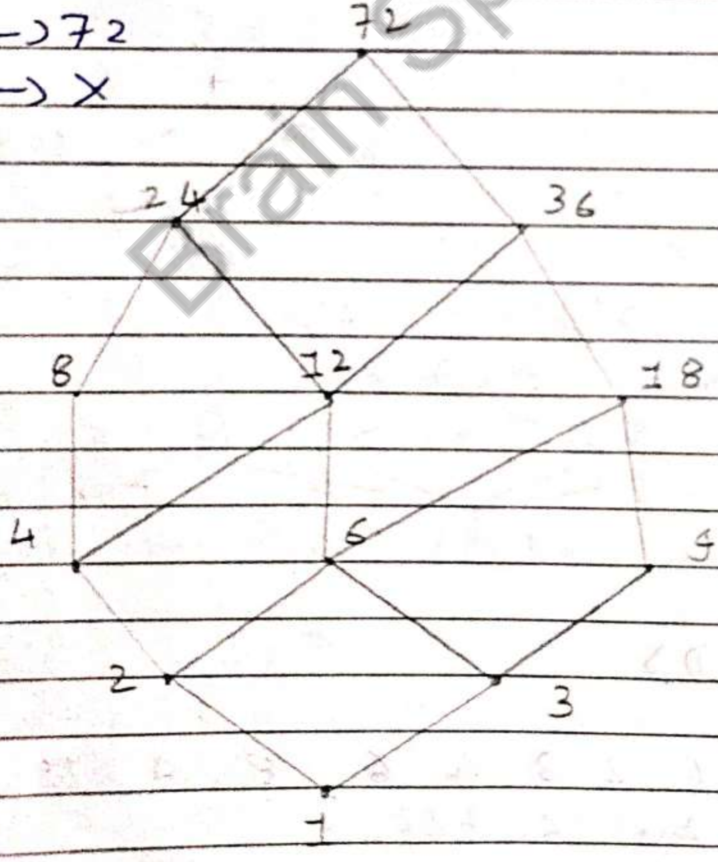
Cover of the all element:

1 -> 2, 3
2 -> 4, 6
3 -> 6, 9
4 -> 8, 12
6 -> 12, 18
8 -> 24
9 -> 18
12 -> 24, 36
18 -> 36
24 -> 72
36 -> 72
72 -> X

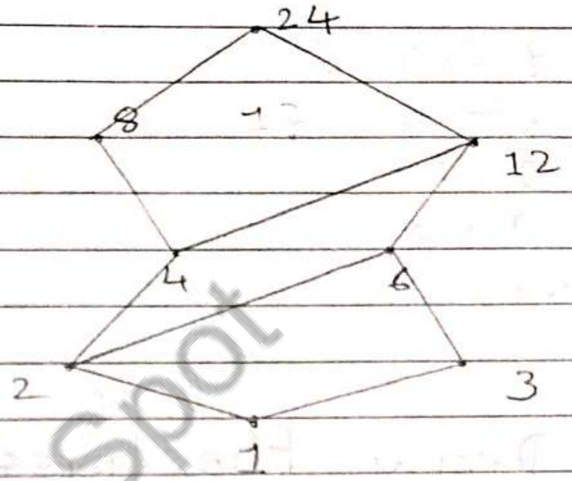


(g) $\langle S_{24}, D \rangle$

$S_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

Cover of all the element:

$1 \rightarrow 2, 3$
$2 \rightarrow 4, 6$
$3 \rightarrow 6$
$4 \rightarrow 8, 12$
$6 \rightarrow 12$
$8 \rightarrow 24$
$12 \rightarrow 24$
$24 \rightarrow X$

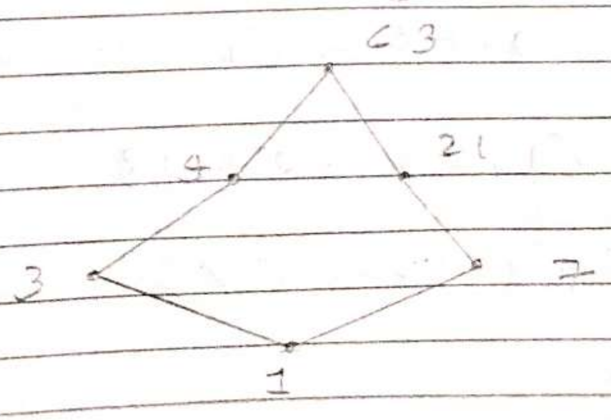


(h) $\langle S_{63}, D \rangle$

$S_{63} = \{1, 3, 7, 9, 21, 63\}$

Cover of all the element:

$1 \rightarrow 3, 7$
$3 \rightarrow 9$
$7 \rightarrow 21$
$9 \rightarrow 63$
$21 \rightarrow 63$
$63 \rightarrow X$



(i) $\langle S_{81}, D \rangle$

$S_{81} = \{1, 3, 9, 27, 81\}$

Cover of the all element:

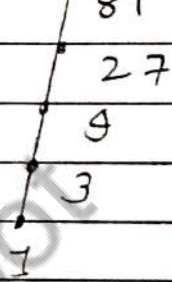
$1 \rightarrow 3$

$3 \rightarrow 9$

$9 \rightarrow 27$

$27 \rightarrow 81$

$81 \rightarrow X$



2 Draw the hass diagram of $\langle L^2, \leq_2 \rangle$; where $L^2 = L \times L$, $L = \{0, 1\}$.

$L^2 = \{\{0,0\}, \{0,1\}, \{1,0\}, \{1,1\}\}$

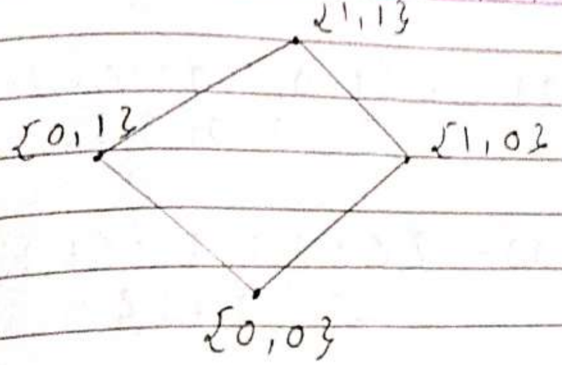
Cover of the all element:

$\{0,0\} \rightarrow \{0,1\}, \{1,0\}$

$\{0,1\} \rightarrow \{1,1\}$

$\{1,0\} \rightarrow \{1,1\}$

$\{1,1\} \rightarrow \{X\}$



3 Draw the Hass Diagram of $\langle L^4, \leq_4 \rangle$ where $L^4 = L \times L \times L \times L$ and $L = \{0,1\}$.

$L^4 = \{(0,0,0,0), (0,0,1,0), (0,1,0,0), (0,1,1,0), (1,0,0,0), (1,0,1,0), (1,1,0,0), (1,1,1,0), (0,0,0,1), (0,0,1,1), (0,1,0,1), (0,1,1,1), (1,0,0,1), (1,0,1,1), (1,1,0,1), (1,1,1,1)\}$

Cover of all the element:

$(0,0,0,0) \to \{(0,0,1,0), (0,0,0,1), (1,0,0,0), (0,1,0,0)\}$

$(0,0,1,0) \to \{(0,1,1,0), (1,0,1,0), (0,0,1,1)\}$

$(0,1,0,0) \to \{(0,1,1,0), (0,1,0,1), (1,1,0,0)\}$

$(0,1,1,0) \to \{(0,1,1,1), (1,1,1,0)\}$

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(1, 0, 0, 0) → {(1, 0, 1, 0), (1, 1, 0, 0), (1, 0, 0, 1)}

(0, 1, 0, 0) → {(0, 1, 1, 0), (1, 1, 0, 0), (0, 1, 0, 1)}

(1, 0, 1, 0) → {(1, 1, 1, 0), (1, 0, 1, 1)}

(0, 0, 1, 1) → {(0, 1, 1, 1), (1, 0, 1, 1)}

(0, 1, 0, 1) → {(0, 1, 1, 1), (1, 1, 0, 1)}

(1, 0, 0, 1) → {(1, 0, 1, 1), (1, 1, 0, 1)}

(1, 1, 0, 0) → {(1, 1, 1, 0), (1, 1, 0, 1)}

(0, 1, 1, 1) → (1, 1, 1, 1)

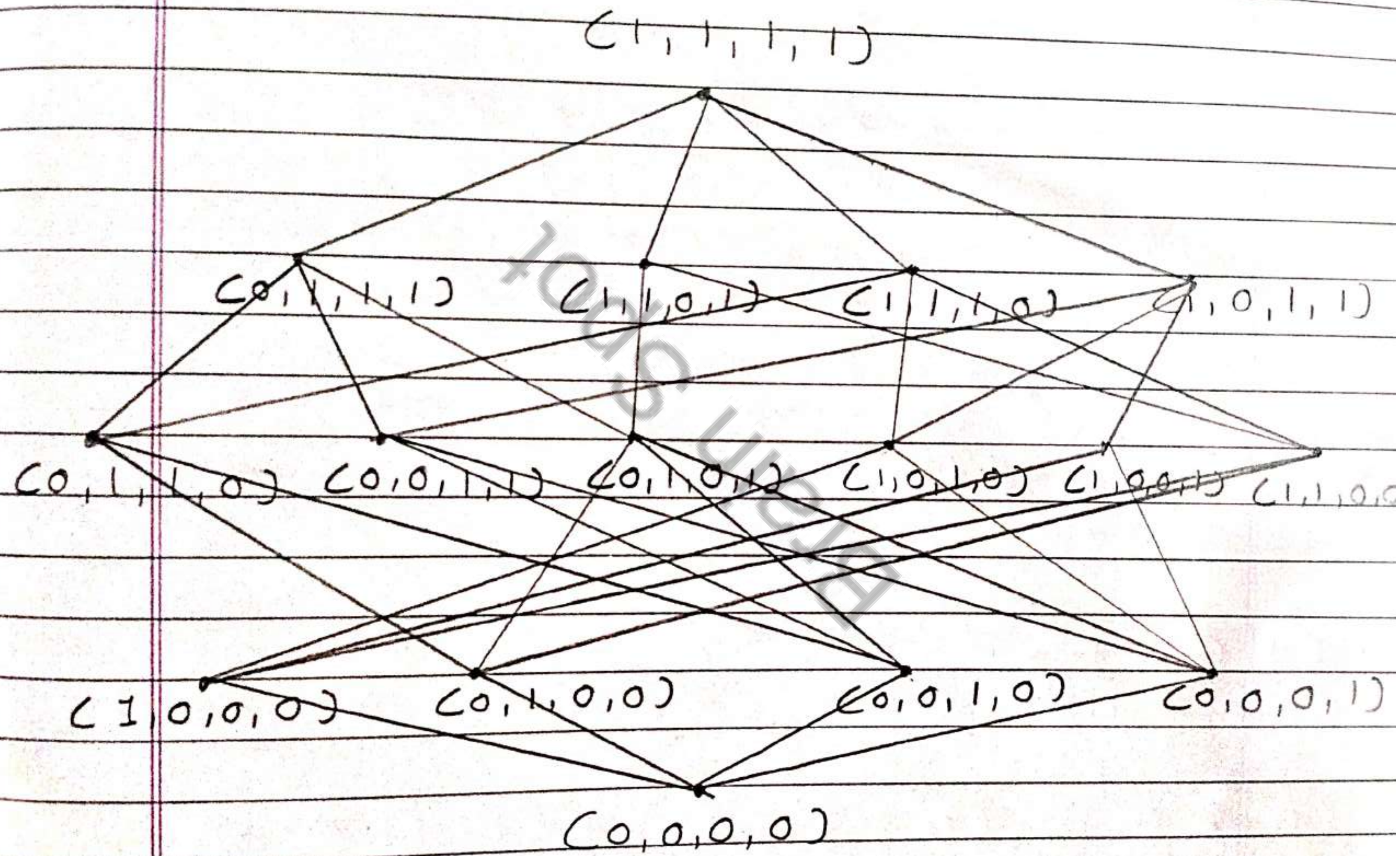
(1, 1, 0, 1) → (1, 1, 1, 1)

(1, 1, 1, 0) → (1, 1, 1, 1)

(1, 0, 1, 1) → (1, 1, 1, 1)

(1, 1, 1, 1) → X

(0, 0, 0, 1) → {(1, 0, 0, 1), (1, 0, 1, 0), (0, 1, 0, 1)}


(1,1,1,1)
(0,1,1,1)
(1,1,0,1)
(1,1,1,0)
(1,0,1,1)
(0,1,1,0)
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(1,0,0,0)
(0,1,0,0)
(0,0,1,0)
(0,0,0,1)
(0,0,0,0)

* Task : 3

1 Define Properties of Lattice.

Let L be a non-empty set closed under two binary operation called meet and join denoted by $\wedge$ and $\vee$.

Then L is called a lattice, if the following Properties are follow.

(1) Commutative Law:

$$a \wedge b = b \wedge a$$
$$a \vee b = b \vee a$$

(2) Associative Law:

$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$
$$(a \vee b) \vee c = a \vee (b \vee c)$$

(3) Absorption Law:

$$a \wedge (a \vee b) = a$$
$$a \vee (a \wedge b) = a$$

2 Give an example of poset which in not lattice.

=> Consider $\langle \{1, 3, 4, 24, 36, 72\}, D \rangle$ is Poset.

$24, 36 \subset \{1, 3, 4, 24, 36, 72\}$

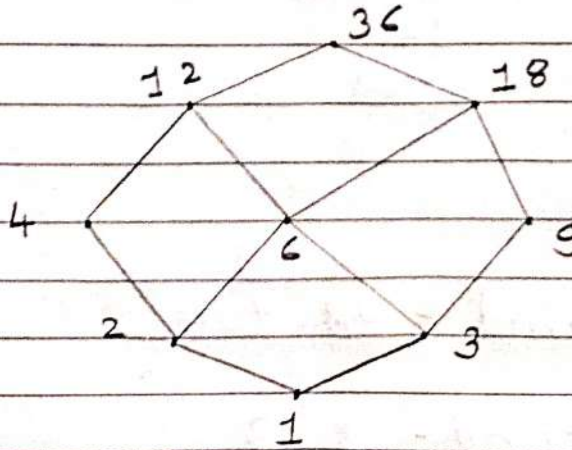
For lattice is follow this three Properties.

1 Commutative law:

$24 * 36 = GCD\{24, 36\}$
$= 12 \notin \{1, 3, 4, 24, 36, 72\}$

So, $\langle \{1, 3, 4, 24, 36, 72\}, D \rangle$ is not lattice.

3 Let $P = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ with divisibility relation Find GLB and LUB of $\{3, 9\}$ and $\{2, 3\}$



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=) For $\{3, 9\}$

$GLB\{3, 9\} = \cancel{9}\ 3$

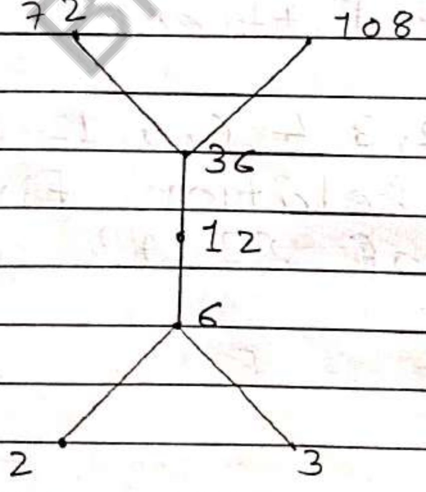
$\cancel{LUB}$ LCM $\{3, 9\} = \cancel{3}\ 9$

=) For $\{2, 3\}$

$\cancel{GLB}$ GCD $\{2, 3\} = \cancel{6}\ 1$

$\cancel{LUB}$ LCM $\{2, 3\} = \cancel{1}\ 6$

4 Let $A = \{1, 2, 3, 6, 12, 36, 72, 108\}$ with divisibility relation. Find upper bound, lower bound, GLB, LUB, maximal and minimal elements.

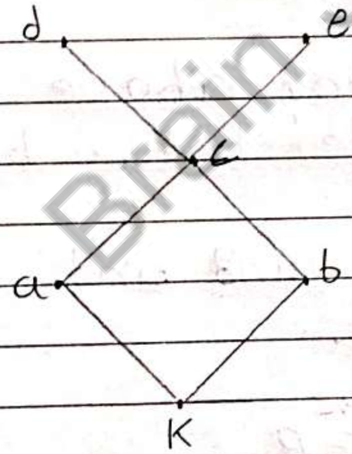


- Upper Bound - 72, 108

- Lower Bound - 2, 3

- Minimal - 2, 3
- Maximal - 72, 108
- GLB - GCD -
- LUB - LCM -

5 Let $A = \{K, a, b, c, d, e\}$ be ordered and represent by hase diagram. If $A = \{K, a, b\}$ then find upper bound and LUB of A.



6 Give an example of complete lattice which is an infinite lattice.

Let $a, b \in R$

$L = [a, b] = \{x \in R \,/\, a \leq x \leq b\}$ is an infinite lattice.

Let $S \subseteq L$, $S \neq \emptyset$

$x \in S \Rightarrow x \in L$
$\Rightarrow a \leq x \leq b$

$\therefore$ S is bounded above by b and S in bounded below by a.

Since, R has LUB and GLB property.

$\therefore$ ~~LUB~~ LCM, $S \in R$
$\therefore a \leq LUBS \leq b$

$\therefore LUB, S \in L$

$\therefore GLB, S \in R$
$\therefore a \leq GLB\, S \leq b$
$\therefore GLB\, S \in L$

$\therefore L = [a, b]$ is a infinite lattice which is complete.

7 Show that $\langle P(x), \subseteq \rangle$ is a bounded lattice where $X = \{a, b, c\}$.

$X = \{a, b, c\}$

$P(x) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

$\forall \ \emptyset \subseteq a \subseteq X$

$\therefore a \subseteq P(x)$

$\therefore GCB(P(x)) = 1$

$\therefore \cancel{LUB}^{LCM}(P(x)) = x$

-> 0 element of $GCB(L) = 0$

$\therefore$ 0 element of $\langle P(x), \subseteq \rangle = \emptyset \in P(x)$

-> 1 element of $\cancel{LUB}^{LCM}(L) = 1$ e

$\therefore$ 1 element of $\langle P(x), \subseteq \rangle = X \in P(x)$

Here, 0 and 1 elements are in $P(x)$.

So, $P(x)$ is a bounded lattice.

8 Show that $\langle S_{30}, D \rangle$ is a bounded lattice.

→ Here $S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$1 \; D \; 1 \; D \; 30$

$1 \; D \; 2 \; D \; 30$

$1 \; D \; 3 \; D \; 30$

$1 \; D \; 5 \; D \; 30$

$1 \; D \; 6 \; D \; 30$

$1 \; D \; 10 \; D \; 30$

$1 \; D \; 15 \; D \; 30$

$1 \; D \; 30 \; D \; 30$

$\therefore 1 \; D \; x \; D \; 30$

$\therefore \forall x \in S_{30}$

$\therefore GCD(S_{30}) = 1$

LCM

$\therefore \cancel{LUB}(S_{30}) = 30$

0 - element of $(S_{30}, D) = 1$

1 - element of $(S_{30}, D) = 30$

$\therefore S_{30}$ is a bounded lattice.

* Task : 4

3 Show that $\langle S_{1001}, D \rangle$ is a lattice.

=> We know that $\langle S_n, D \rangle$ is a Poset $\forall n \in N$.

$\therefore \langle S_{1001}, D \rangle$ is a Poset.

$S_{1001} = \{1, 7, 13, 11, 77, 91, 143, 1001\}$

-> $a * b = a \wedge b =$ ~~GLB~~ GCD

$a * b$	1	7	11	13	77	91	143	1001
1	1	1	1	1	1	1	1	1
7	1	7	1	1	7	1	1	1
11	1	1	11	1	11	1	13	11
13	1	1	1	13	1	7	11	13
77	1	7	11	1	77	1	11	77
91	1	1	1	17	7	91	13	91
143	1	1	13	11	11	13	143	143
1001	1	7	77	13	77	91	143	1001

-> $a \vee b =$ ~~LUB~~ LCM

a V b	1	7	11	13	77	91	143	1001
1	1	7	11	13	77	91	143	1001
7	7	7	77	91	77	91	1001	1001
11	11	77	11	143	77	1001	143	1001
13	13	91	143	13	1001	91	143	1001
77	77	77	77	1001	1001	1001	1001	1001
91	91	91	1001	1001	1001	1001	1001	1001
143	143	1001	143	1001	1001	143	143	1001
1001	1001	1001	1001	1001	1001	1001	1001	1001

From the table,

$a \vee b \in S_{1001}$, $a \wedge b \in S_{1001}$.

$\therefore \langle S_{1001}, D \rangle$ is a lattice.

4 Show that $S = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$ is a sub lattice of $\langle P(A), \cap, \cup \rangle$ for $A = \{a, b, c\}$

-> We know that $\langle P(A), \cap, \cup \rangle$ is Poset.

$A = \{a, b, c\}$

$P(A) = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \emptyset, \{a,b,c\}\}$

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$S = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

Let $a, b \in P(A)$

→ For $a \wedge b$ = GLB.

$\wedge$	ϕ	$\{a\}$	$\{b\}$	$\{a,b\}$
ϕ	ϕ	ϕ	ϕ	ϕ
$\{a\}$	ϕ	$\{a\}$	ϕ	$\{a\}$
$\{b\}$	ϕ	ϕ	$\{b\}$	$\{b\}$
$\{a,b\}$	ϕ	$\{a\}$	$\{b\}$	$\{a,b\}$

→ For $a \vee b$ = ~~LUB~~ LCM

$\vee$	ϕ	$\{a\}$	$\{b\}$	$\{a,b\}$
ϕ	ϕ	$\{a\}$	$\{b\}$	$\{a,b\}$
$\{a\}$	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$
$\{b\}$	$\{b\}$	$\{a,b\}$	$\{b\}$	$\{a,b\}$
$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$

$\therefore$ $a \vee b \in P(x)$, $a \wedge b \in P(x)$

$\therefore$ So, S is a sub lattice of $\langle P(A), \cap, \vee \rangle$ where
$A = \{a, b, c\}$.

5 Show that $S = \{1, 2, 3, 6\}$ is a sub lattice of $\langle S_{30}, GCD, LCM \rangle$

$S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$S = \{1, 2, 3, 6\}$

→ For $a \wedge b = GCD$

GCD	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

→ For $a \vee b = LCM$

LCM	1	2	3	6
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

$\therefore a \wedge b \in S_{30}$, $a \vee b \in S_{30}$

$\therefore$ So, S is a sub lattice of the S_{30}.

1 Find complement of each element of Lattice $\langle S_{10}, GCD, LCM, 1, 10 \rangle$.

2 Find Complement of each element of Lattice $\langle S_{30}, GCD, LCM, 1, 30 \rangle$