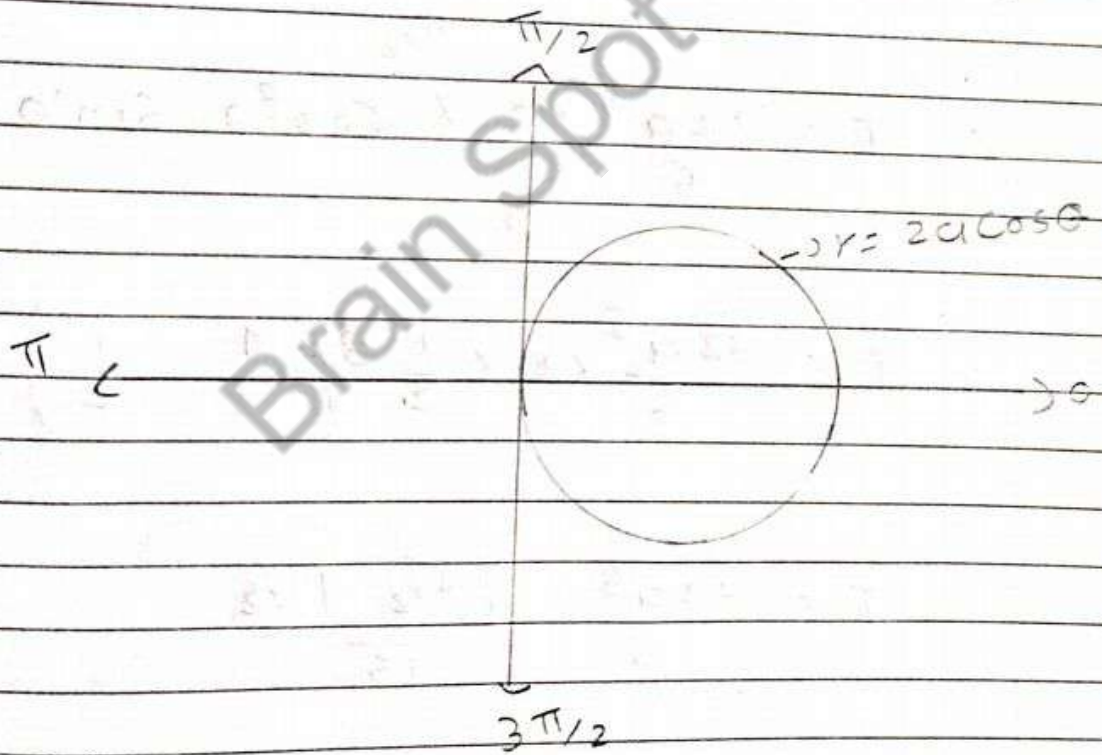


* Task - 5 : Evaluation to Double Integrals over a given Region.

1 Evaluate $\iint_R r^4 \cos^3 \theta \, dr \cdot d\theta$ over the interior of the circle $r = 2a \cos \theta$.

Here $I = \iint_R r^4 \cos^3 \theta \, dr \cdot d\theta$ Integral bounded by Region $r = 2a \cos \theta$ Circle.



$$\text{Here } I = \int_{\theta = -\pi/2}^{\pi/2} \int_{r=0}^{2a \cos \theta} r^4 \cdot \cos^3 \theta \, dr \cdot d\theta$$

$$\therefore I = \int_{\theta = -\pi/2}^{\pi/2} \cos^3 \theta \left[\frac{r^5}{5} \right]_0^{2a \cos \theta} \cdot d\theta$$

$$\therefore I = \int_{\theta = -\pi/2}^{\pi/2} \cos^3 \theta \frac{(2a \cos \theta)^5}{5} \cdot d\theta$$

$$\therefore I = \frac{32a^5}{5} \int_{-\pi/2}^{\pi/2} \cos^5 \theta \cdot d\theta \cos^3 \theta$$

$$\therefore I = \frac{32a^5 \times 2}{5} \int_0^{\pi/2} \cos^4 \theta \cdot \sin \theta \cdot d\theta$$

$$\therefore I = \frac{32a^5}{5} \times 2 \times \frac{1}{2} B\left(\frac{9}{2}, \frac{1}{2}\right)$$

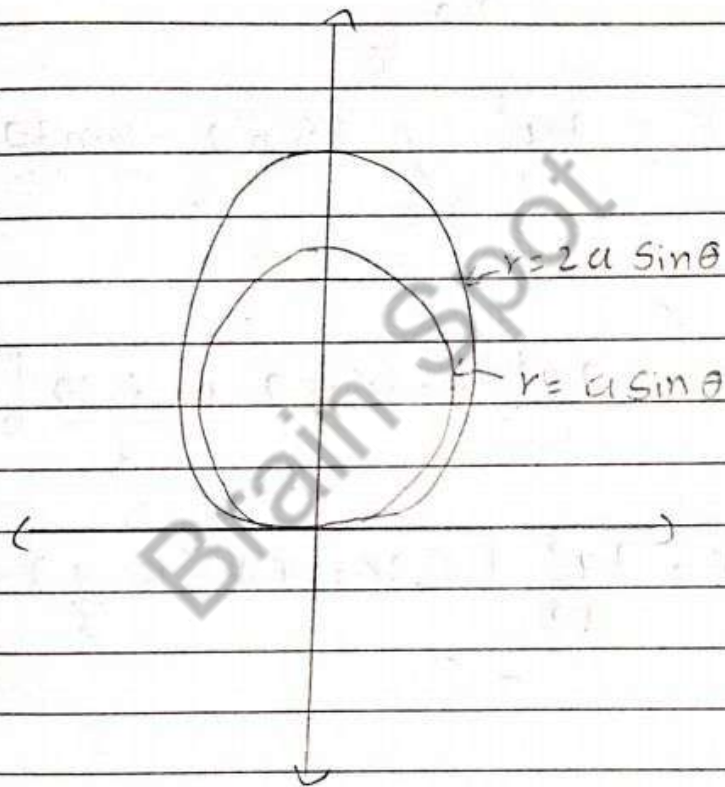
$$\therefore I = \frac{32a^5}{5} \frac{\Gamma(9/2) \cdot \Gamma(1/2)}{\Gamma(5)}$$

$$\therefore I = \frac{32a^5}{5} \times \frac{7/2 \times 5/2 \times 3/2 \times 1/2 \times \pi}{5 \times 4 \times 3 \times 2}$$

$$\therefore I = \frac{7\pi a^5}{4}$$

2 Evaluate $\iint_R r^2 dr \cdot d\theta$ over the area between $r = a \sin \theta$ and $r = 2a \sin \theta$.

Here $I = \iint_R r^2 dr \cdot d\theta$



$$I = \int_{\theta=0}^{\pi} \int_{r=a \sin \theta}^{2a \sin \theta} r^2 dr \cdot d\theta$$

$$\therefore I = \int_{\theta=0}^{\pi} \left[\frac{r^3}{3} \right]_{a \sin \theta}^{2a \sin \theta} \cdot d\theta$$

$$\therefore I = \int_{\theta=0}^{\pi} \frac{(2a \sin \theta)^3 - (a \sin \theta)^3}{3} \cdot d\theta$$

$$\therefore I = \int_{\theta=0}^{\pi} \frac{8a^3 \sin^3 \theta - a^3 \sin^3 \theta}{3} \cdot d\theta$$

$$\therefore I = \frac{7a^3}{3} \int_{\theta=0}^{\pi} \sin^3 \theta \cdot d\theta$$

$$\therefore I = \frac{7a^3}{3} \int_{\theta=0}^{\pi} \frac{3 \sin \theta - \sin 3\theta}{4} \cdot d\theta$$

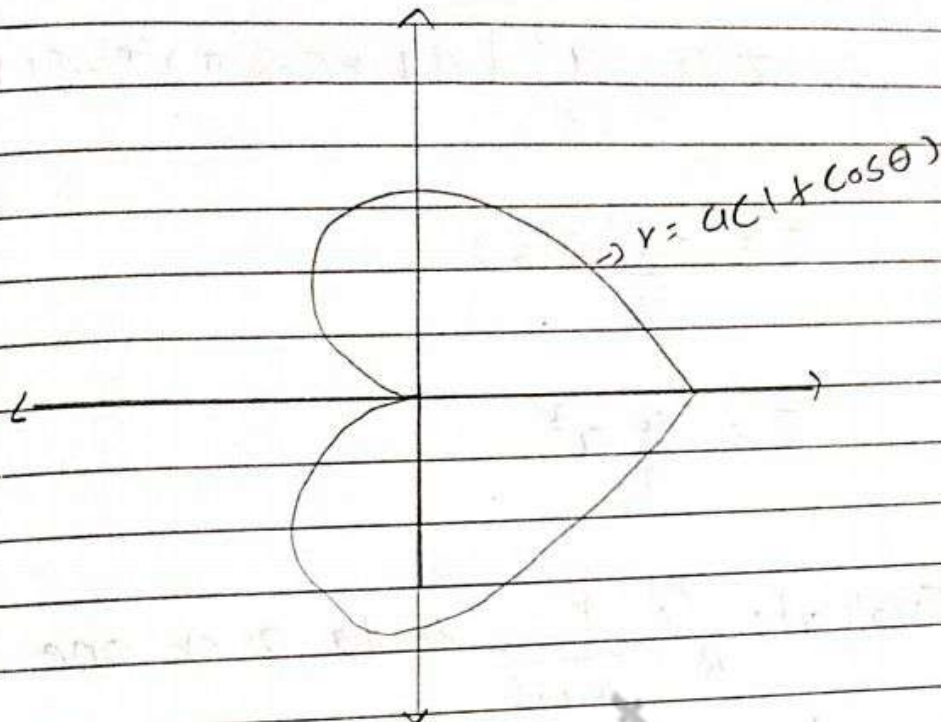
$$\therefore I = \frac{7a^3}{12} \left[-3 \cos \theta + \frac{\cos 3\theta}{3} \right]_0^{\pi}$$

$$\therefore I = \frac{7a^3}{12} \left[(-3 \cos \pi + \frac{\cos 3\pi}{3}) - (-3 \cos 0 + \frac{\cos 3(0)}{3}) \right]$$

$$\therefore I = \frac{28a^3}{9}$$

3 Evaluate $\iint_R r^2 \cdot \sin \theta \, dr \cdot d\theta$ over the cardioid $r = a(1 + \cos \theta)$ above the initial line.

Here, $I = \iint_R r^2 \sin \theta \, dr \cdot d\theta$



$$\therefore I = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r^2 \cdot \sin\theta \, dr \cdot d\theta$$

$$\therefore I = \int_{\theta=0}^{\pi} \sin\theta \left[\frac{r^3}{3} \right]_0^{a(1+\cos\theta)} d\theta$$

$$\therefore I = \int_{\theta=0}^{\pi} \sin\theta \frac{(a(1+\cos\theta))^3}{3} \cdot d\theta$$

$$\therefore I = \frac{-a^3}{3} \int_{\theta=0}^{\pi} (-\sin\theta) (1+\cos\theta)^3 \cdot d\theta$$

$$\therefore I = \frac{-a^3}{3} \left[\frac{(1+\cos\theta)^4}{4} \right]_0^{\pi}$$

$$\therefore I = \frac{-a^3}{12} \left[(1 + \cos \pi)^4 - (1 + \cos 0)^4 \right]$$

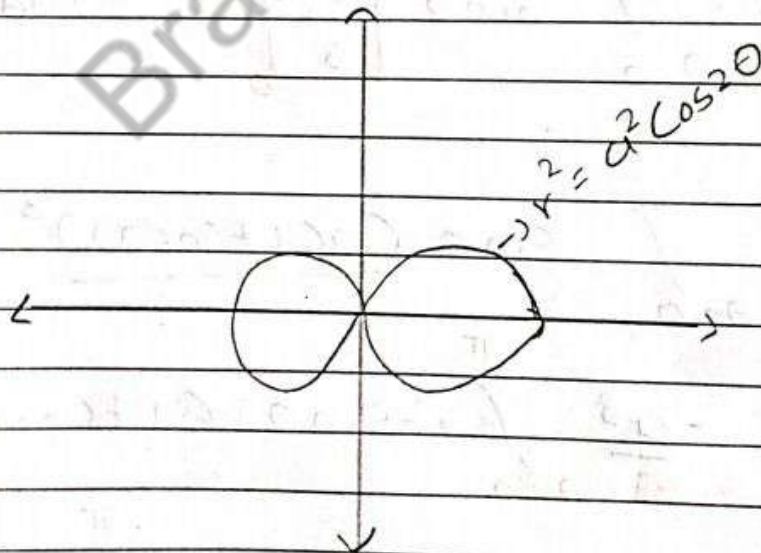
$$\therefore I = \frac{-a^3}{12} (-16)$$

$$\therefore I = \frac{4}{3} a^3$$

4 Evaluate $\iint_R \frac{r}{\sqrt{a^2+r^2}} dr \cdot d\theta$ over one loop of

Lemniscate $r^2 = a^2 \cos 2\theta$

$$\text{Here, } I = \iint_R \frac{r}{\sqrt{a^2+r^2}}$$



$$\therefore I = \int_{\theta = -\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{r=0}^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{a^2+r^2}} dr \cdot d\theta$$

$$\therefore I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{a \cos 2\theta} \frac{1}{2} \frac{2r}{\sqrt{a^2+r^2}} dr \cdot d\theta$$

$$\therefore I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} \left[2(r^2+a^2)^{\frac{1}{2}} \right]_0^{a \cos 2\theta} \cdot d\theta$$

$$\therefore I = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2a((\cos 2\theta + 1)^{\frac{1}{2}} - 1) \cdot d\theta$$

$$\therefore I = \frac{2a}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2\cos 2\theta - 1) \cdot d\theta$$

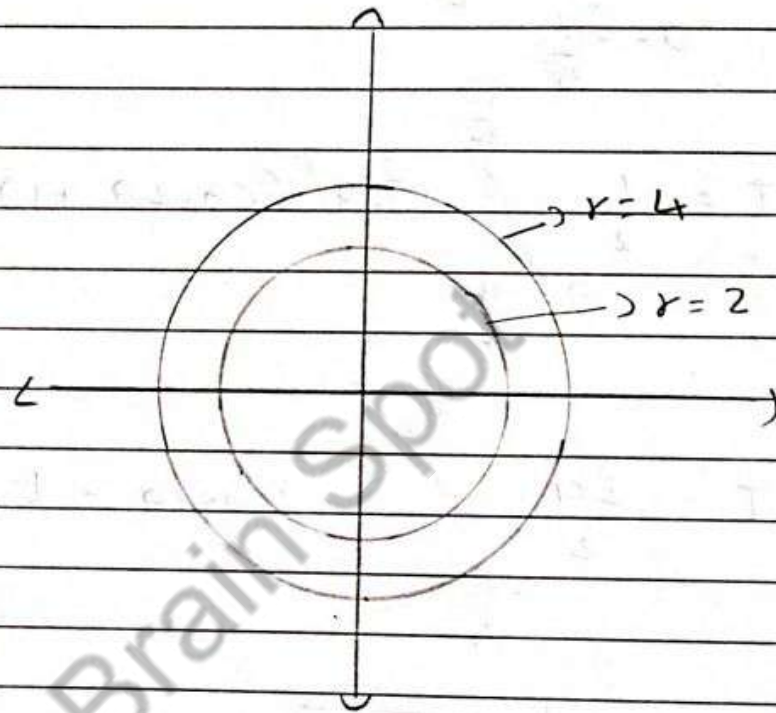
$$\therefore I = a \left[\sqrt{2} \sin \theta - \theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$\therefore I = a \left[\sqrt{2} \sin \frac{\pi}{4} - \frac{\pi}{4} - \sqrt{2} \sin \left(-\frac{\pi}{4} \right) + \left(-\frac{\pi}{4} \right) \right]$$

$$\therefore I = a \left[\sqrt{2} \sin \frac{\pi}{4} - \frac{\pi}{4} - \sqrt{2} \sin \left(-\frac{\pi}{4} \right) + \left(-\frac{\pi}{4} \right) \right]$$

$$\therefore I = a \left(2 - \frac{\pi}{2} \right)$$

5 Evaluate $\iint_R r^3 \sin 2\theta \, dr \, d\theta$ over the area bounded in First quadrant R between the circles $r=2$ and $r=4$.



$$\text{Here, } I = \int_{\theta=0}^{\pi/2} \int_{r=2}^4 r^3 \sin 2\theta \, dr \, d\theta$$

$$\therefore I = \int_{\theta=0}^{\pi/2} \sin 2\theta \left[\frac{r^4}{4} \right]_2^4 \, d\theta$$

$$\therefore I = \int_{\theta=0}^{\pi/2} \sin 2\theta [64 - 4] \, d\theta$$

$$\therefore T = 60 \int_{\theta=0}^{\pi/2} 2 \sin \theta \cdot \cos \theta \cdot d\theta$$

$$\therefore T = 120 \int_0^{\pi/2} \sin \theta \cdot \cos \theta \cdot d\theta$$

$$\therefore T = 120 \times \frac{1}{2} \beta(1, 1)$$

$$\therefore T = 60$$

Brain Spot

$$\therefore I = \frac{1}{12}$$

$$2 \int_0^{\pi/2} \int_{a(1-\cos\theta)}^a r^2 dr \cdot d\theta$$

$$\text{Here, } I = \int_{\theta=0}^{\pi/2} \int_{r=a(1-\cos\theta)}^a r^2 dr \cdot d\theta$$

$$\therefore I = \int_{\theta=0}^{\pi/2} \left[\frac{r^3}{3} \right]_{a(1-\cos\theta)}^a d\theta$$

$$\therefore I = \int_{\theta=0}^{\pi/2} \frac{1}{3} [(a)^3 - (a(1-\cos\theta))^3] \cdot d\theta$$

$$\therefore I = \frac{a^3}{3} \int_0^{\pi/2} (1 - (1-\cos\theta))^3 d\theta$$

$$\therefore I = \frac{a^3}{3} \int_0^{\pi/2} (1 - 1 + 3\cos\theta - 3\cos^2\theta + \cos^3\theta) d\theta$$

$$\therefore I = \frac{a^3}{3} \int_0^{\pi/2} 3\cos\theta \cdot \sin^0\theta - 3\cos^2\theta \cdot \sin^0\theta + \cos^3\theta \cdot \sin^0\theta \cdot d\theta$$

$$\therefore I = \frac{a^3}{3} \left[\frac{3}{2} \beta \left(\frac{1}{2}, 1 \right) - \frac{3}{2} \beta \left(\frac{1}{2}, \frac{3}{2} \right) + \frac{1}{2} \beta \left(\frac{1}{2}, 2 \right) \right]$$

$$\therefore I = \frac{a^3}{3} \left[\frac{3}{2} \frac{\Gamma_{1/2} \Gamma_{1/2}}{\Gamma_{3/2}} - \frac{3}{2} \frac{|\Gamma_{1/2}|^{3/2}}{\Gamma_2} + \frac{1}{2} \frac{|\Gamma_{1/2}|^{2 \times 2}}{\Gamma_{5/2}} \right]$$

$$\therefore I = \frac{a^3}{3} \left[\frac{3 - 3\pi + 2}{4} \right]$$

$$\therefore I = \frac{a^3}{3} \left[\frac{11 - 3\pi}{4} \right]$$

$$3 \int_0^{\pi/2} \int_0^{\pi/2} \sin(\theta + \phi) d\theta \cdot d\phi$$

$$\text{Here, } I = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \sin(\theta + \phi) d\phi \cdot d\theta$$

$$\therefore I = \int_{\theta=0}^{\pi/2} \left[-\cos(\theta + \phi) \right]_0^{\pi/2} \cdot d\theta$$

$$\therefore I = \int_{\theta=0}^{\pi/2} -\cos\left(\frac{\pi}{2} + \theta\right) - \cos\theta \cdot d\theta$$

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$$\therefore I = - \int_0^{\pi/2} -\sin \theta - \cos \theta \cdot d\theta$$

$$\therefore I = - \left[\cos \theta - \sin \theta \right]_0^{\pi/2}$$

$$\therefore I = - \left[\cos \frac{\pi}{2} - \sin \frac{\pi}{2} - \cos 0 + \sin 0 \right]$$

$$\therefore I = 2$$

$$4 \int_0^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2} dr \cdot d\theta$$

$$\text{Here, } I = \int_{\theta=0}^{\pi/4} \int_{r=0}^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2} dr \cdot d\theta$$

$$\therefore I = \frac{1}{2} \int_{\theta=0}^{\pi/4} \int_{r=0}^{\sqrt{\cos 2\theta}} \frac{2r}{(1+r^2)^2} dr \cdot d\theta$$

$$\therefore I = \frac{1}{2} \int_0^{\pi/4} \left[-(1+r^2)^{-1} \right]_0^{\sqrt{\cos 2\theta}} \cdot d\theta$$

$$\therefore I = -\frac{1}{2} \int_0^{\pi/4} \frac{1}{1+\cos 2\theta} - 1 \cdot d\theta$$

$$\therefore I = -\frac{1}{2} \int_0^{\pi/4} \frac{1}{2} \sec^2 \theta - 1 \cdot d\theta$$

~~$$\therefore I = -\frac{1}{2} \int_0^{\pi}$$~~

$$\therefore I = -\frac{1}{2} \left[\frac{1}{2} \tan \theta - \theta \right]_0^{\pi/4}$$

$$\therefore I = -\frac{1}{2} \left(\frac{1}{2} \tan \frac{\pi}{4} - \frac{\pi}{4} \right)$$

$$\therefore I = \frac{1}{8} (\pi - 2)$$

$$5 \int_0^a \int_0^{\cos^{-1}\left(\frac{r}{a}\right)} r \sin \theta \, dr \cdot d\theta$$

~~Here $I = \int_{\theta=0}^{\cos^{-1}\left(\frac{r}{a}\right)} \int_{r=0}^a r \sin \theta \, dr \cdot d\theta$~~

~~Here $I = \int_{r=0}^a \int_{\theta=0}^{\cos^{-1}\left(\frac{r}{a}\right)} r \sin \theta \, d\theta \cdot dr$~~

$$\therefore I = \int_{r=0}^a r \left[-\cos \theta \right]_0^{\cos^{-1}\left(\frac{r}{a}\right)} \cdot dr$$

$$\therefore I = \int_{r=0}^a r \left(\frac{-\cos\left(\frac{r}{a}\right)}{\frac{1}{a}} + \cos\theta \right) \cdot dr$$

$$\therefore I = \int_{r=0}^a r \left(-\frac{r}{a} + 1 \right) \cdot dr$$

$$\left(\because \cos(\cos^{-1}x) = x \right)$$

$$\therefore I = \int_{r=0}^a r \left(\frac{r^2}{a} \right) dr$$

$$\therefore I = \left[\frac{r^2}{2} - \frac{r^3}{3a} \right]_0^a$$

$$\therefore I = \frac{a^2}{2} - \frac{a^3}{3a}$$

$$\therefore I = \frac{a^2}{6}$$

$$6 \int_{-\pi/2}^{\pi/2} \int_0^{a \sin\theta} r^2 dr \cdot d\theta$$

$$\text{Here } I = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{a \sin\theta} r^2 dr \cdot d\theta$$

$$\therefore I = \int_{\theta = -\pi/2}^{\pi/2} \left[\frac{r^3}{3} \right]_0^{a \sin \theta} \cdot d\theta$$

$$\therefore I = \int_{\theta = -\pi/2}^{\pi/2} \frac{a^3 \sin^3 \theta}{3} \cdot d\theta$$

$$\therefore I = \frac{a^3}{3} (0)$$

(Here, $f(x) \cdot dx = 0$ Because $\sin \theta$ is odd function)

$$\therefore I = 0$$

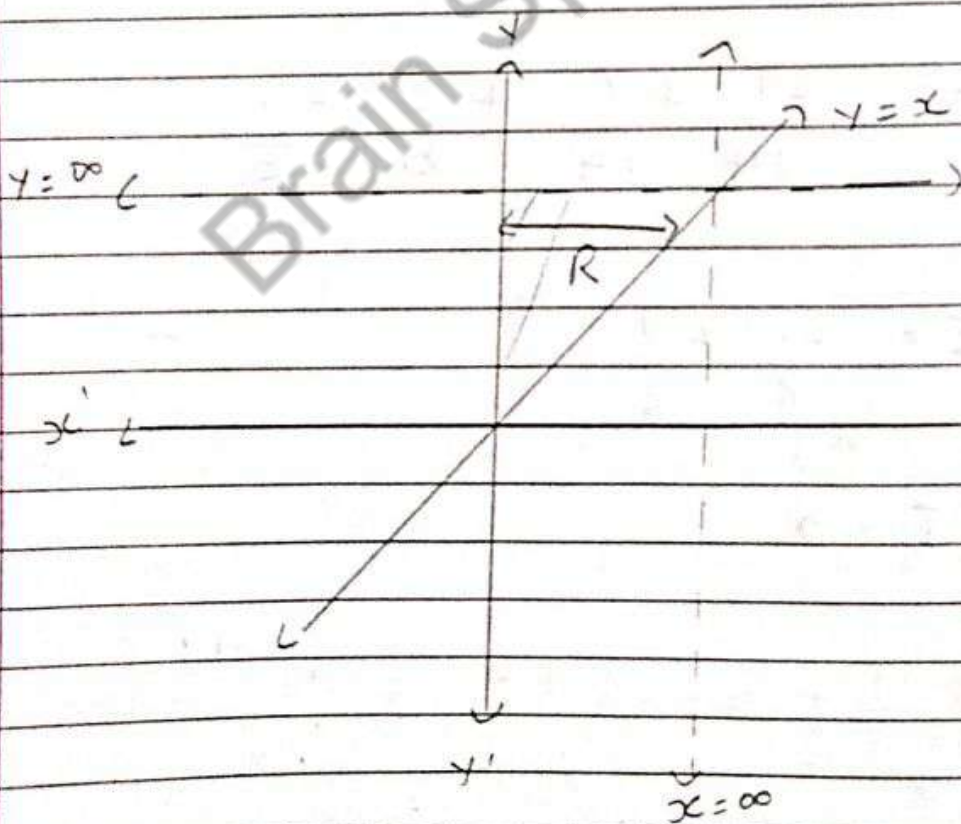
* Task : 3 Change the order of Integration

$$1 \int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} dA$$

Here, we want to evaluate the

$$I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y} dy dx$$

We have, $x=0$, $x=\infty$, $y=x$, $x=\infty$



$$\Rightarrow \int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} dy dx = \int_{y=0}^{\infty} \int_{x=0}^y \frac{e^{-y}}{y} dx dy$$

$$I = \int_{y=0}^{\infty} \frac{e^{-y}}{y} [x]_0^y \cdot dx$$

$$\therefore I = \int_{y=0}^{\infty} e^{-y} dx$$

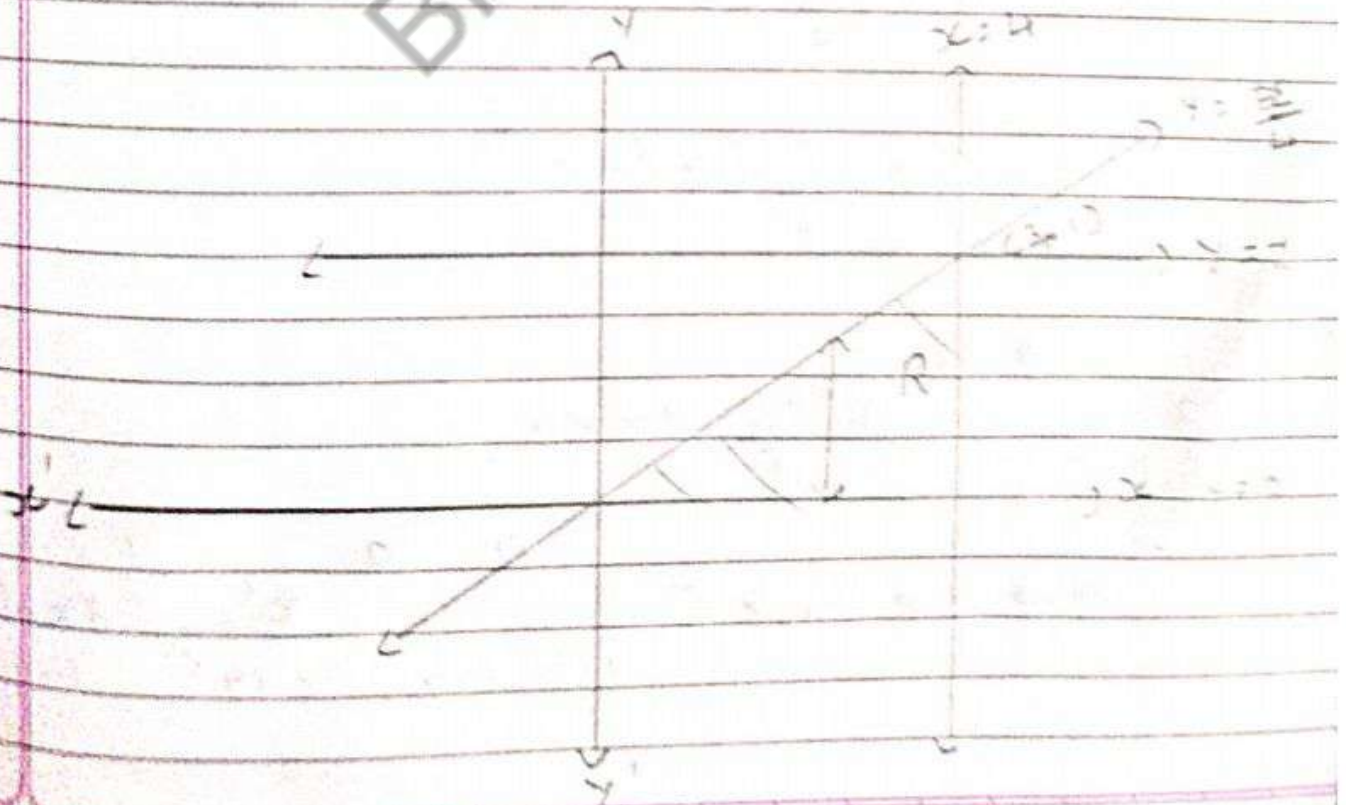
$$\therefore I = [e^{-y}]_0^{\infty}$$

$$\therefore I = 1$$

$$2 \int_0^1 \int_{4y}^4 e^{x^2} dx dy$$

$$\text{Here } I = \int_{y=0}^1 \int_{x=4y}^4 e^{x^2} dx dy$$

We have, $y=0$, $y=1$, $x=4$ and $x=4y$



$$\Rightarrow \int_0^4 \int_{4y}^4 e^{x^2} dx dy = \int_{x=0}^4 \int_{y=0}^{x/4} e^{x^2} dy dx$$

$$I = \int_{x=0}^4 e^{x^2} \cdot [y]_0^{x/4} dx$$

$$\therefore I = \int_{x=0}^4 \frac{x \cdot e^{x^2}}{4} dx$$

$$\therefore I = \frac{1}{8} \int_{x=0}^4 e^{x^2} \cdot 2x \cdot dx$$

$$\therefore I = \frac{1}{8} [e^{x^2}]_0^4$$

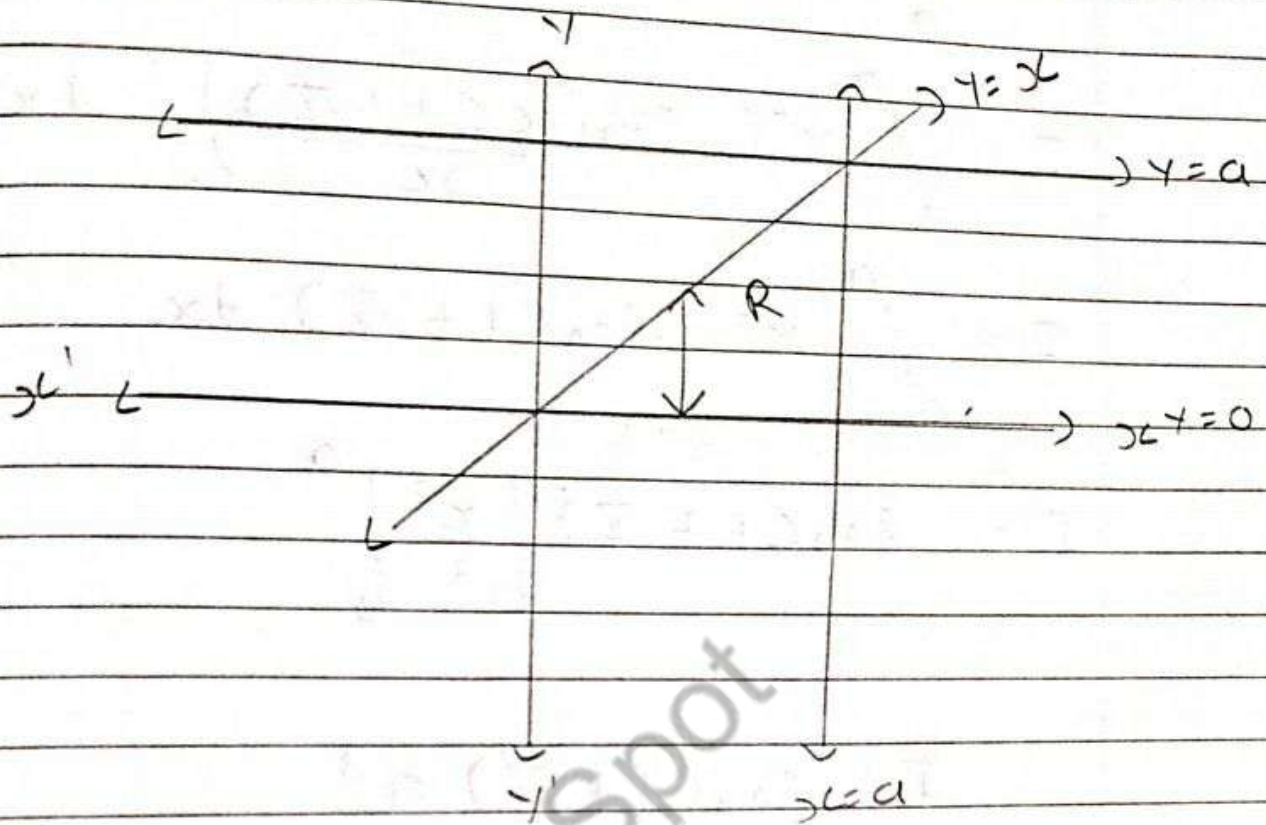
$$\therefore I = \frac{1}{8} [e^{16} - e^0]$$

$$\therefore I = \frac{1}{8} [e^{16} - 1]$$

3 $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} dx dy$

Here we have $I = \int_{y=0}^a \int_{x=y}^a \frac{x^2}{\sqrt{x^2+y^2}} dx dy$

We have $y=0$, $y=a$, $y=x$, $x=a$



$$\Rightarrow \int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} dx dy = \int_{x=0}^a \int_{y=0}^x \frac{x^2}{\sqrt{x^2+y^2}} dy dx$$

$$\therefore I = \int_{x=0}^a \int_{y=0}^x x^2 \cdot \frac{1}{\sqrt{x^2+y^2}} dy dx$$

$$\therefore I = \int_{x=0}^a \left[x^2 \log(y + \sqrt{x^2+y^2}) \right]_0^x dx$$

$$\therefore I = \int_0^a x^2 \left[\log(x + \sqrt{x^2+x^2}) - \log(\sqrt{x^2}) \right] dx$$

$$I = \int_0^a x^2 [\log(x + x\sqrt{2}) - \log x] dx$$

$$I = \int_0^a x^2 \left(\log \left(\frac{x(1 + \sqrt{2})}{x} \right) \right) dx$$

$$I = \int_0^a x^2 \cdot \log(1 + \sqrt{2}) dx$$

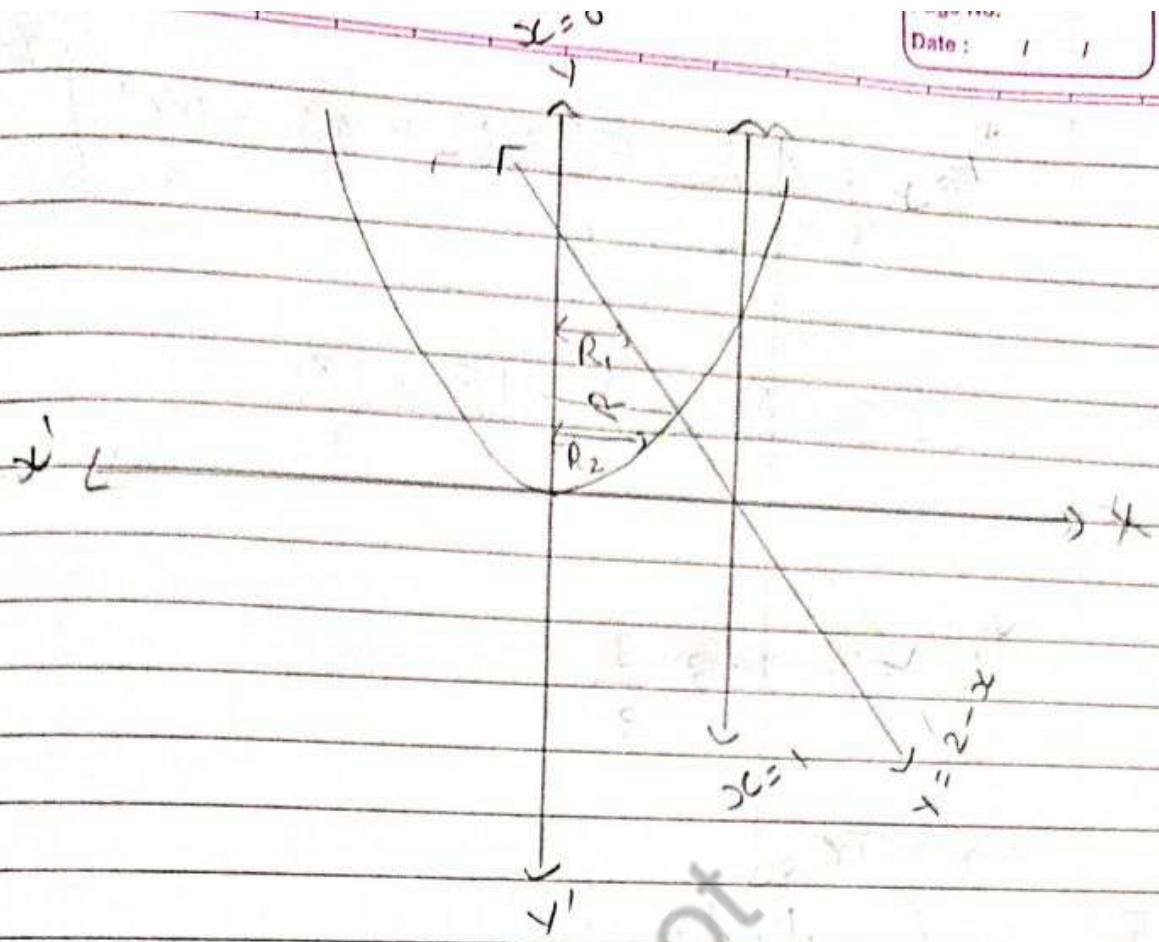
$$I = \log(1 + \sqrt{2}) \left[\frac{x^3}{3} \right]_0^a$$

$$\therefore I = \log(1 + \sqrt{2}) \frac{a^3}{3}$$

4 $\int_0^1 \int_{x^2}^{2-x} xy \, dA$

Here $I = \int_{x=0}^1 \int_{y=x^2}^{2-x} xy \, dy \cdot dx$

We have $x=0$, $x=1$, $y=x^2$ and $y=2-x$



$$\Rightarrow \int_{x=0}^1 \int_{y=x^2}^{2-x} xy \, dy \cdot dx = \int_{y=0}^2 \int_{x=0}^y xy \, dx \, dy$$

$$I = \iint_{R_1} xy \, dy \, dx + \iint_{R_2} xy \, dy \, dx$$

$$\therefore I = \int_{y=0}^1 \int_{x=0}^{\sqrt{y}} xy \, dx \, dy + \int_{y=1}^2 \int_{x=0}^{2-y} xy \, dx \, dy$$

$$\therefore I = \int_0^1 y \left[\frac{x^2}{2} \right]_0^{\sqrt{y}} dy + \int_1^2 y \left[\frac{x^2}{2} \right]_0^{2-y} dy$$

$$\therefore I = \int_0^1 \frac{y^2}{2} \cdot dy + \int_1^2 y \frac{(4 - 4y + y^2)}{2} dy$$

$$\therefore I = \left[\frac{y^3}{6} \right]_0^1 + \left[\frac{2y^2}{2} + \frac{y^4}{8} - \frac{4y^3}{6} \right]_0^2$$

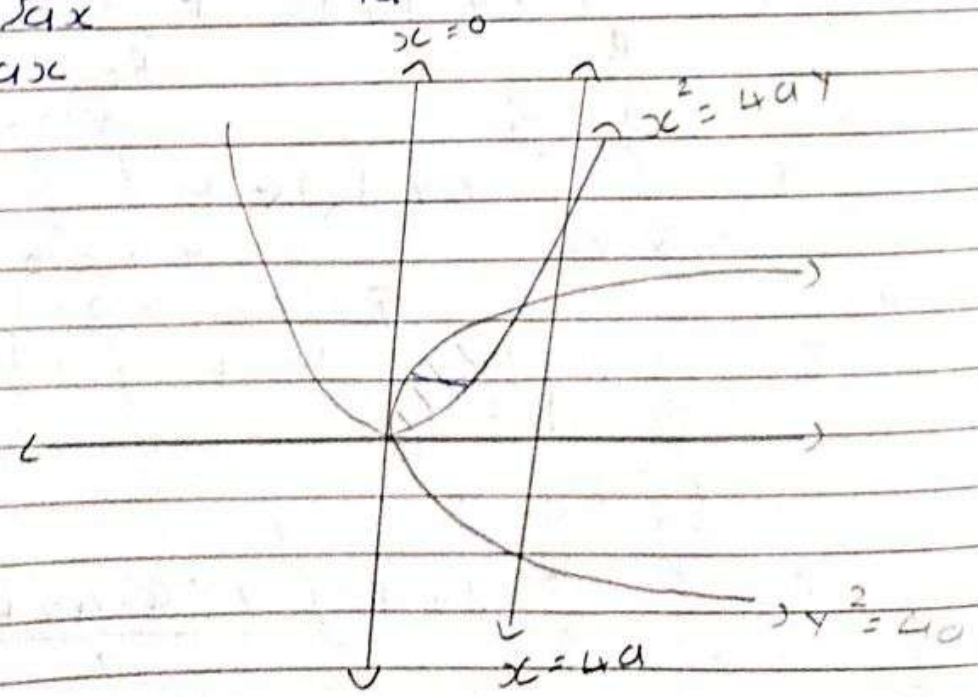
$$\therefore I = \frac{1}{6} + 3 - \frac{15}{8} + \frac{18}{3}$$

$$\therefore I = 318 \frac{3}{8}$$

5 $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy dx$

Here, $I = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} 1 dy dx$

$y = 2\sqrt{ax}$
 $y^2 = 4ax$



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$$\Rightarrow \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} 1 \, dy \, dx = \int_{y=0}^{4a} \int_{x=y^2/4a}^{2\sqrt{ay}} 1 \, dx \, dy$$

$$I = \int_{y=0}^{4a} \left[x \right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$\therefore I = \int_{y=0}^{4a} 2\sqrt{ay} - \frac{y^2}{4a} \cdot dy$$

$$\therefore I = \int_0^{4a} 2\sqrt{a} y^{1/2} - \frac{1}{4a} y^2 \cdot dy$$

$$\therefore I = \left[2\sqrt{a} \frac{y^{3/2}}{3/2} - \frac{1}{4a} \frac{y^3}{3} \right]_0^{4a}$$

$$\therefore I = \frac{4\sqrt{a}}{3} \cdot (a)^{3/2} - \frac{1}{4a} \frac{(4a)^3}{3}$$

$$\therefore I = \frac{4a \cdot a^3}{3} - \frac{1}{4a} \frac{(64a^3)}{3}$$

$$\therefore I = \frac{32a^2 - 16a^2}{3}$$

$$\therefore I = \frac{16a^2}{3}$$

$$6 \int_0^{1/2} \int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}} dx dy$$

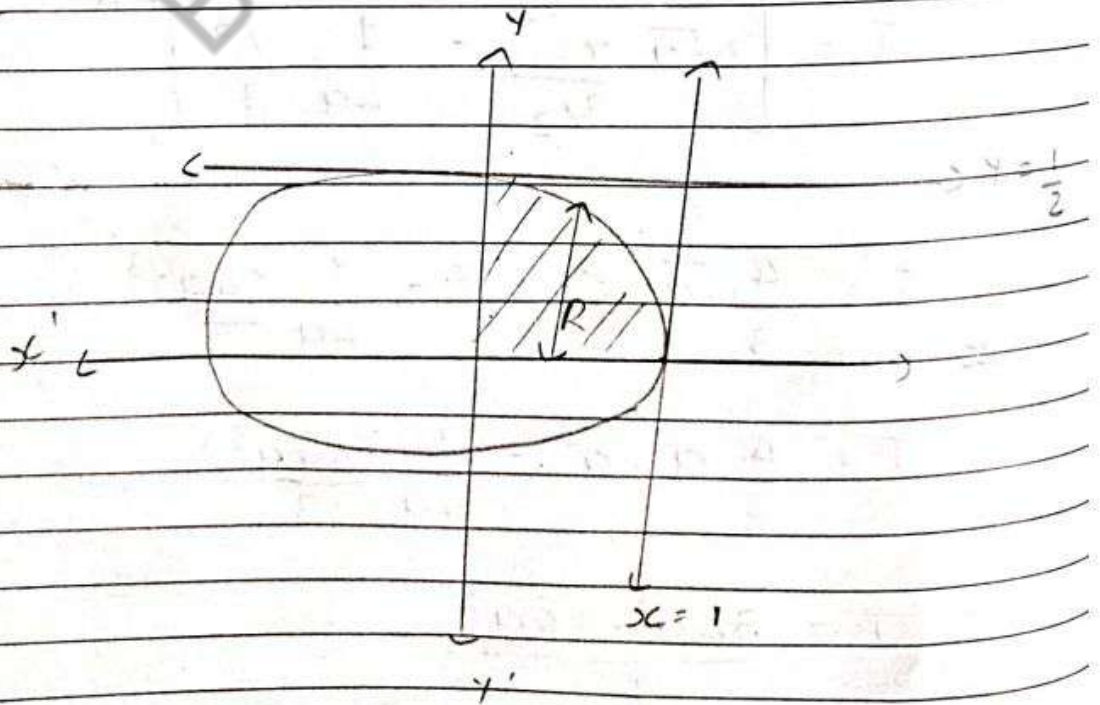
Here $I = \int_{y=0}^{1/2} \int_{x=0}^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}} dx dy$

We have $y=0, y=1/2$
 $x=0, x=\sqrt{1-4y^2}$

$$\therefore x^2 = 1 - 4y^2$$

$$\therefore x^2 + 4y^2 = 1$$

$$\therefore \frac{x^2}{(1)^2} + \frac{y^2}{(1/2)^2} = 1$$



$$I = \int_{x=0}^1 \int_{y=0}^{\frac{1}{2}\sqrt{1-x^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}} dy dx$$

$$I = \int_{x=0}^1 \frac{1+x^2}{\sqrt{1-x^2}} \int_{y=0}^{\frac{1}{2}\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2-y^2}} dy dx$$

$$I = \int_{x=0}^1 \frac{1+x^2}{\sqrt{1-x^2}} \left[\sin^{-1} \left(\frac{y}{\sqrt{1-x^2}} \right) \right]_0^{\frac{1}{2}\sqrt{1-x^2}} dx$$

$$I = \int_{x=0}^1 \frac{1+x^2}{\sqrt{1-x^2}} \left(\sin^{-1} \left(\frac{1}{2} \right) - \sin^{-1}(0) \right) dx$$

$$I = \int_0^1 \frac{\pi}{6} \cdot \frac{2(1-x^2)}{\sqrt{1-x^2}} dx$$

$$I = \frac{\pi}{6} \int_0^1 \frac{2}{\sqrt{1-x^2}} - \frac{(1-x^2)}{\sqrt{1-x^2}} dx$$

$$I = \frac{\pi}{6} \left[2 \sin^{-1}(x) + \frac{1}{2} \sin^{-1}(1) \right]_0^1$$

$$= \frac{\pi}{6} \left(\frac{3}{2} \sin^{-1}(1) \right)$$

$$= \frac{\pi}{8}$$

તીવ્ર ઇચ્છાથી પથ્થરની દિવાલને પણ તોડી શકાય છે.

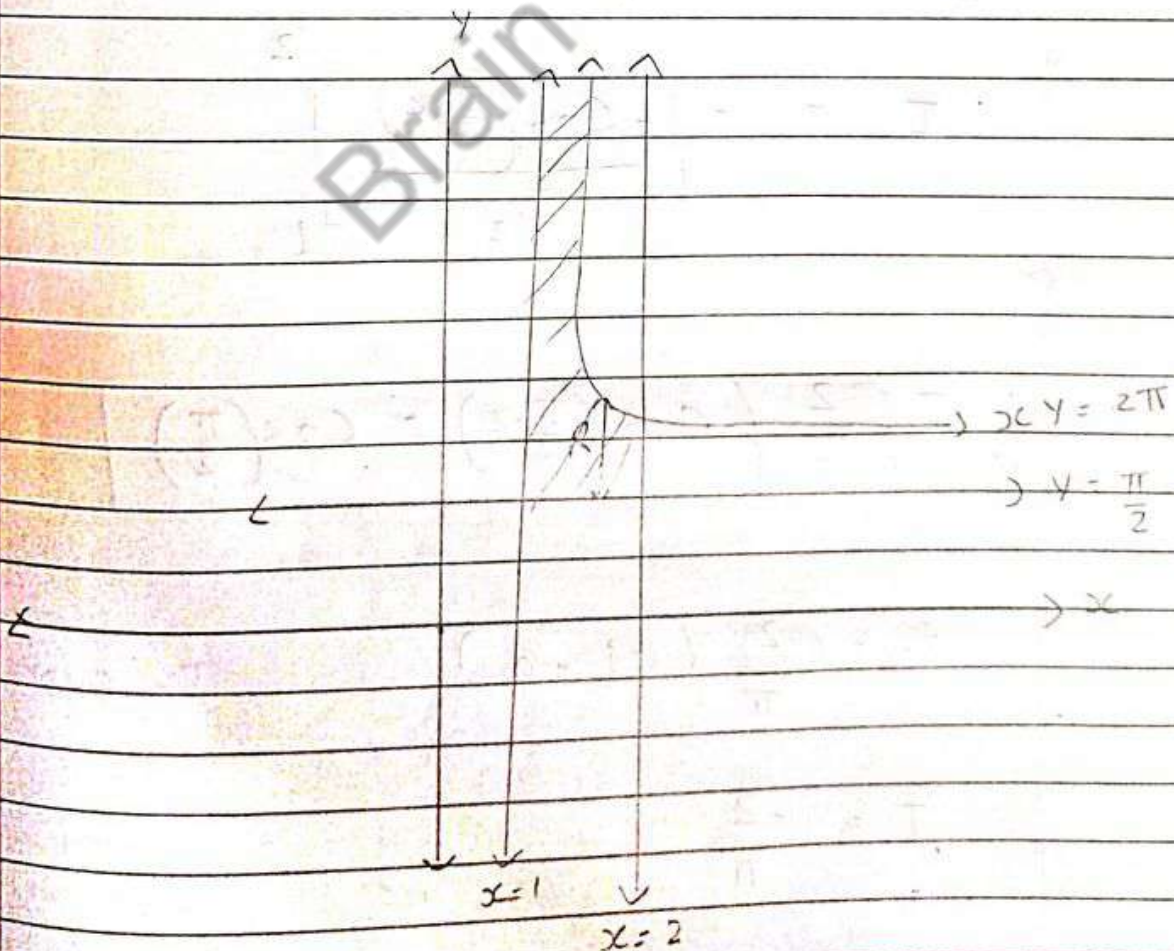
* Task : 2 Calculate the Integral over a Given region.

1 Evaluate $\iint_R x \cos(xy) dA$, where R is

the Region in closed by $x=1$, $x=2$, $y=\frac{\pi}{2}$ and $y=2\pi$.

Here, we want to evaluate $\iint_R x \cos(xy) dA$ where R is region,

$x=1$, $x=2$, $y=\frac{\pi}{2}$ and $y=2\pi$



તીવ્ર ઘસ્યાથી પથ્થરની દિવાલને પણ તોડી શકાય છે.

$$\text{Here } I = \int_{x=1}^2 \int_{y=\frac{\pi}{2}}^{\frac{2\pi}{x}} x \cos(xy) dy dx$$

$$\therefore I = \int_{x=1}^2 x \left[\frac{\sin(xy)}{x} \right]_{\frac{\pi}{2}}^{\frac{2\pi}{x}} dx$$

$$\therefore I = \int_{x=1}^2 \sin\left(x \cdot \frac{2\pi}{x}\right) - \sin\left(x \cdot \frac{\pi}{2}\right) dx$$

$$\therefore I = \int_{x=1}^2 \sin(2\pi) - \sin\left(\frac{\pi}{2}x\right) dx$$

$$\therefore I = - \left[\frac{-\cos\left(\frac{\pi}{2}x\right)}{\pi/2} \right]_1^2$$

$$\therefore I = \frac{2}{\pi} \left(\cos\left(\frac{\pi}{2} \cdot 2\right) - \cos\left(\frac{\pi}{2}\right) \right)$$

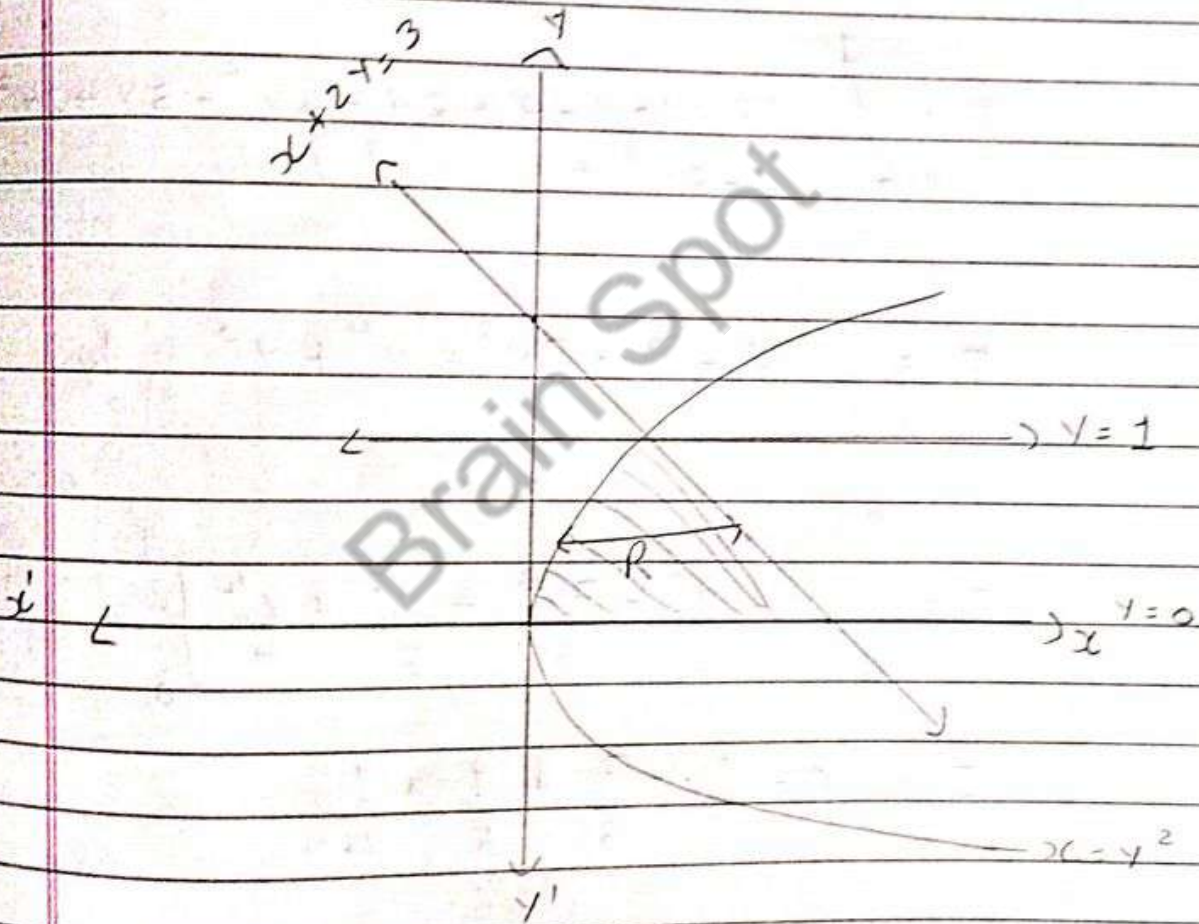
$$\therefore I = \frac{2}{\pi} (-1 - 0)$$

$$\therefore I = \frac{-2}{\pi}$$

2 Evaluate $\iint_R (5 - 2x - y) dy dx$, where R is the region bounded by $y=0$, $x+2y=3$, $x=y^2$.

Here, Region bounded by,

$$y=0, x+2y=3, x=y^2$$



Here
$$I = \int_{y=0}^1 \int_{x=y^2}^{3-2y} (5 - 2x - y) dx dy$$

$$\therefore I = \int_{y=0}^1 \left[5x - \frac{2x^2}{2} - yx \right]_{x=3-2y}^{x=3-2y} dy$$

$$\therefore I = \int_{y=0}^1 \left[5(3-2y) - (3-2y)^2 - y(3-2y) \right] dy$$

$$= \int_{y=0}^1 \left[5y^2 - y^4 - y^3 \right] dy$$

$$\therefore I = \int_{y=0}^1 \left[15 - 10y - 9 + 12y - 4y^2 - 3y + 2y^2 - 5y^2 + y^4 + y^3 \right] dy$$

$$\therefore I = \int_{y=0}^1 \left[6 - y - 7y^2 + y^4 + y^3 \right] dy$$

$$\therefore I = \left[6y - \frac{y^2}{2} - \frac{7y^3}{3} + \frac{y^5}{5} + \frac{y^4}{4} \right]_0^1$$

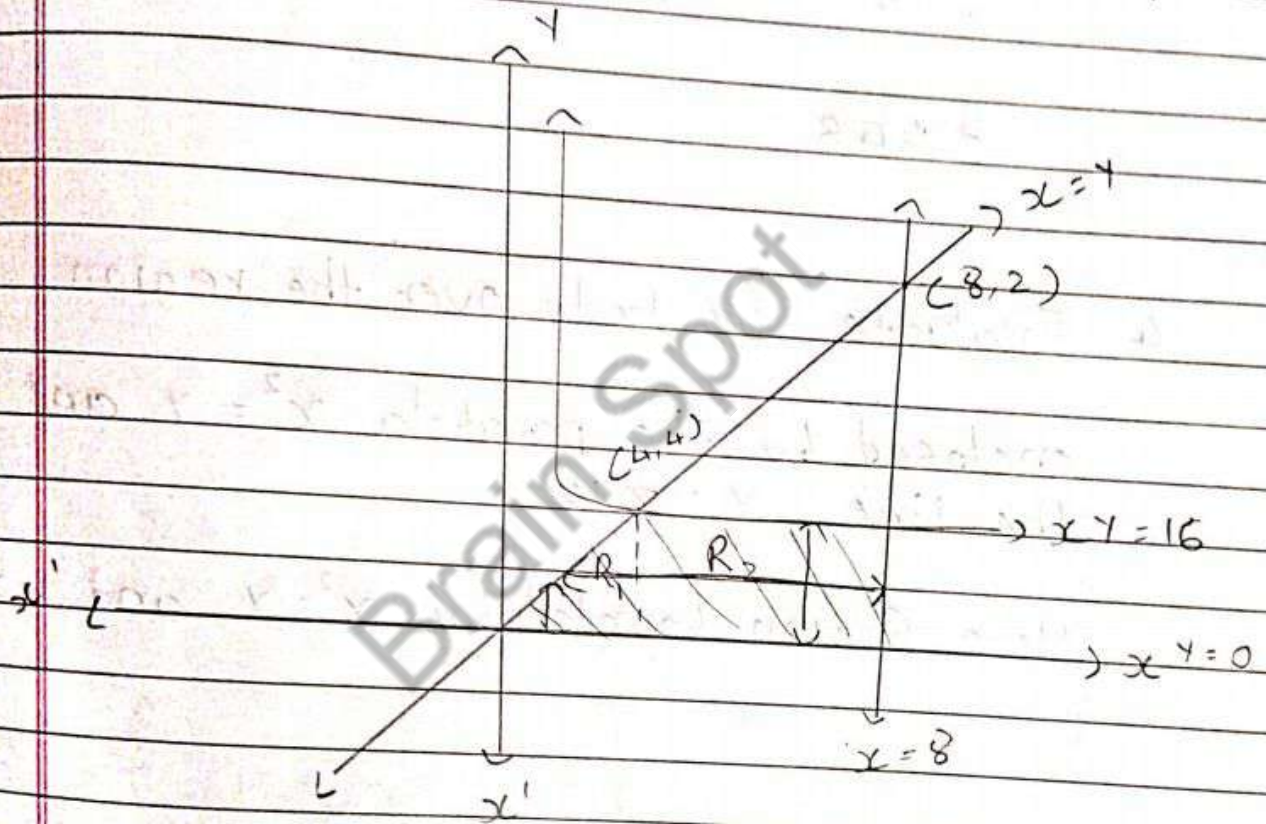
$$\therefore I = 6 - \frac{1}{2} - \frac{7}{3} + \frac{1}{5} + \frac{1}{4}$$

$$\therefore I = \frac{217}{60}$$

3 Evaluate $\iint_R x^2 \cdot dA$, where R is the region bounded by $xy = 16$ lines $y = x$, $y = 0$, $x = 8$ in First quadrant hyperbola.

Here Region bounded by,

$xy = 16$, $y = x$, $y = 0$ and $x = 8$ lines.



Here, $I = \iint_{R_1} x^2 \cdot dA + \iint_{R_2} x^2 \cdot dA$

$$= \int_{x=0}^4 \int_{y=0}^x x^2 dy \cdot dx + \int_{x=4}^8 \int_{y=0}^{16/x} x^2 dy \cdot dx$$

$$= \int_{x=0}^4 x^2 [y]_0^x \cdot dx + \int_{x=4}^8 x^2 [y]_0^{16/x} \cdot dx$$

ત્રીજી ઘટકાથી પથ્થરની દિવાલને પણ તોડી શકાય છે.

$$I = \int_0^4 x^3 dx + \int_4^8 \frac{16}{x} \cdot x^2 dx$$

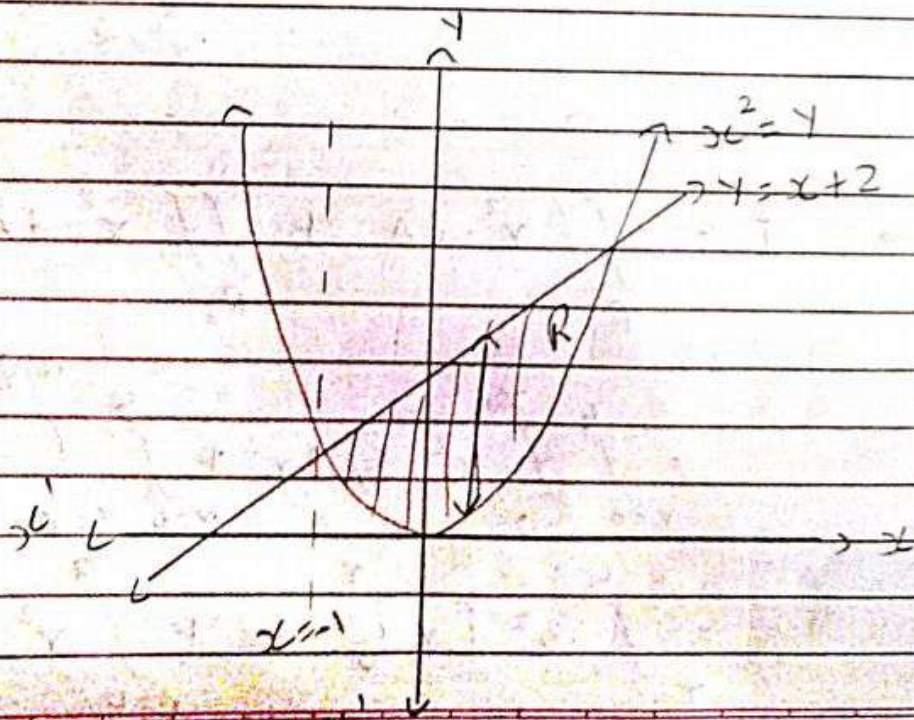
$$I = \left[\frac{x^4}{4} \right]_0^4 + 16 \left[\frac{x^2}{2} \right]_4^8$$

$$I = \frac{(4)^4}{4} + 16 \left(\frac{64}{2} - \frac{16}{2} \right)$$

$$= 448$$

4 Evaluate $\iint y dx dy$ over the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$

Here, Region bounded by $x^2 = y$ and $y = x + 2$



સાથ સાથે શાંતિ માટે સહન કરવું ફરજિયાત છે.

$$\text{Here } I = \int_{x=-1}^2 \int_{y=x^2}^{x+2} y \, dy \, dx$$

$$\therefore I = \int_{x=-1}^2 \left[\frac{y^2}{2} \right]_{x^2}^{x+2} dx$$

$$\therefore I = \int_{x=-1}^2 \frac{1}{2} \left((x+2)^2 - x^4 \right) dx$$

$$\therefore I = \int_{x=-1}^2 \frac{1}{2} (x^2 + 4x + 4 - x^4) dx$$

$$\therefore I = \frac{1}{2} \int_{x=-1}^2 x^2 - x^4 + 4x + 4 \, dx$$

$$\therefore I = \frac{1}{2} \left[\frac{x^3}{3} - \frac{x^5}{5} + 4\frac{x^2}{2} + 4x \right]_{-1}^2$$

$$\therefore I = \frac{1}{2} \left(\frac{8}{3} - \frac{32}{5} + 2(4) + 8 \right) - \left(\frac{-1}{3} + \frac{-1}{5} + 2(-4) \right)$$

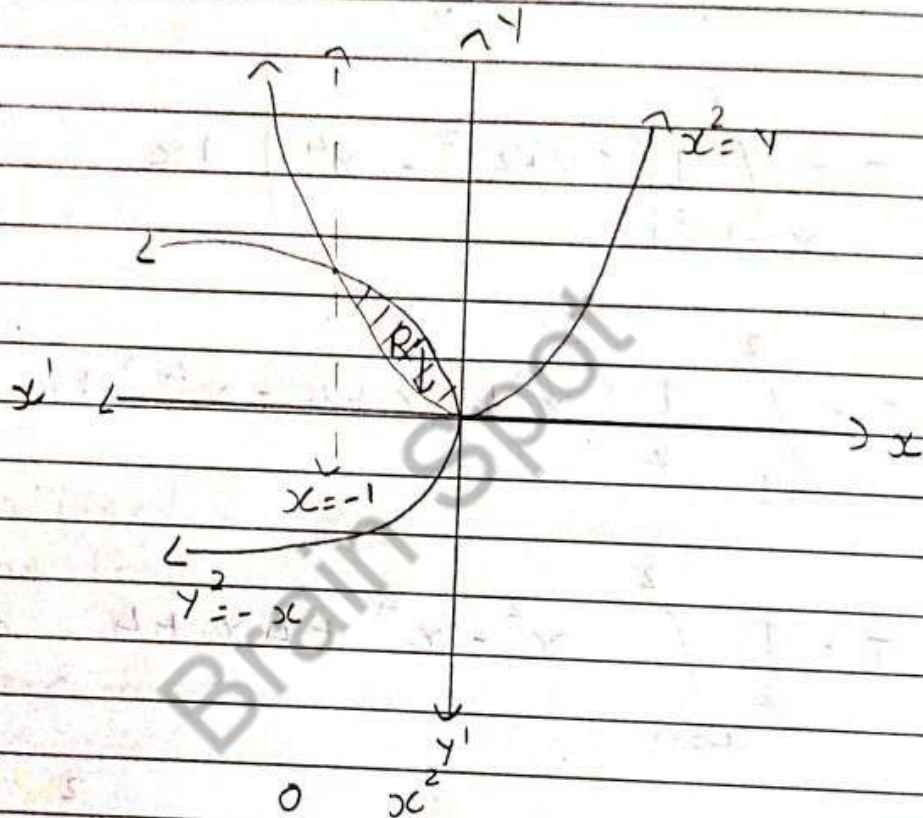
$$\therefore I = \frac{1}{2} \left(\frac{8}{3} - \frac{32}{5} + 8 + 8 \right) + \frac{1}{3} - \frac{1}{5} + 2$$

$$\therefore I = \frac{36}{5}$$

તીવ્ર દષ્ટિથી પચ્ચરની દિવાલને પણ તોડી શકાય છે.

5 Evaluate $\iint xy(x+y) dx dy$ over the region enclosed by the parabolas $x^2 = y$ and $y^2 = -x$

Here, Region bounded by $x^2 = y$ and $y^2 = -x$



Here, $I = \int_{x=-1}^0 \int_{y=-\sqrt{x}}^{\sqrt{x}} x^2 y + xy^2 dy dx$

$$\therefore I = \int_{x=-1}^0 x^2 \left[\frac{y^2}{2} \right]_{-\sqrt{x}}^{\sqrt{x}} + x \left[\frac{y^3}{3} \right]_{-\sqrt{x}}^{\sqrt{x}} dx$$

$$\therefore I = \int_{x=-1}^0 x^2 \left[\frac{x^4}{2} - \frac{x}{2} \right] + x \left[\frac{-x^6}{3} - \frac{(-\sqrt{x})^3}{3} \right] dx$$

$$\therefore I = \int_{x=-1}^0 \left(\frac{x^6}{2} - \frac{x^3}{2} - \frac{x^7}{3} + \frac{x^{5/2}}{3} \right) dx$$

~~$$\therefore I = \int_{-1}^0 x$$~~

~~$$\therefore I = \left[\frac{x^2}{2} \right]$$~~

$$\therefore I = \left[\frac{x^7}{14} - \frac{x^4}{8} - \frac{x^8}{24} + \frac{2x^{7/2}}{21} \right]_{-1}^0$$

$$\therefore I = \frac{-1}{24} + \frac{1}{14} + \frac{2}{21} - \frac{1}{8}$$

$$\therefore I = \frac{-1}{6} + \frac{1}{6}$$

$$\therefore I = 0$$

Unit : 6 Multiple Integrals and its Application

* Task 1: Evaluate the Double Integrals.

$$2 \int_0^1 \int_0^x e^{y/x} \cdot dy \cdot dx$$

Here, $I = \int_0^1 \int_0^x e^{y/x} dy \cdot dx$ given.

$$\therefore I = \int_{x=0}^1 \int_{y=0}^x e^{y/x} dy \cdot dx$$

$$= \int_{x=0}^1 \left[\frac{e^{y/x}}{1/x} \right]_0^x \cdot dx$$

$$= \int_{x=0}^1 x (e^{x/x} - e^0) \cdot dx$$

$$= \int_{x=0}^1 x (e^1 - 1) \cdot dx$$

$$I = \left[\frac{x^2}{2} \right]_0^1 (e^1 - 1)$$

$$I = \frac{1}{2} (e^1 - 1)$$

3 $\int_0^1 \int_0^y x y e^{-x^2} dA$

Here $I = \int_0^1 \int_0^y x e^{-x^2} y \cdot dA$

$$\therefore I = \int_{y=0}^1 \int_{x=0}^y x e^{-x^2} \cdot y \cdot dx dy$$

$$I = -\frac{1}{2} \int_{y=0}^1 \int_{x=0}^y (-2x) e^{-x^2} \cdot y \cdot dx dy$$

$$= -\frac{1}{2} \int_{y=0}^1 \left[e^{-x^2} \right]_0^y \cdot y \cdot dy$$

$$= -\frac{1}{2} \int_{y=0}^1 e^{-y} \cdot y \cdot dy$$

$$= -\frac{1}{4} \int_{y=0}^1 (-2y) \cdot e^{-y} \cdot dy$$

$$I = \frac{1}{4} \left[e^{-y^2} \right]_0^1$$

$$I = \frac{1}{4} e^{-1}$$

$$I = \frac{1}{464e}$$

$$5 \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} dx dy$$

$$\text{Here, } I = \int_{x=0}^1 \int_{y=0}^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} \cdot dy dx$$

$$I = \int_{x=0}^1 \int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2 + y^2} \cdot dy dx$$

$$I = \int_{x=0}^1 \int_{y=0}^{\sqrt{1+x^2}} \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \left(\frac{y}{\sqrt{1+x^2}} \right) \right]_0^{\sqrt{1+x^2}} \cdot dx$$

$$I = \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4} \cdot dx$$

$$I = \frac{\pi}{4} \int_{x=0}^1 \frac{1}{\sqrt{(1)^2 + (x^2)}} \cdot dx$$

$$I = \frac{\pi}{4} \left[\log(1 + \sqrt{1+x^2}) \right]_0^1$$

$$I = \frac{\pi}{4} \log(1 + \sqrt{2})$$

$$1 \int_0^a \int_0^b \frac{1}{xy} dx \cdot dy$$

Here $I = \int_0^a \int_0^b \frac{1}{xy} dx \cdot dy$

$$I = \int_{y=0}^a \int_{x=0}^b \frac{1}{xy} \cdot dx \cdot dy$$

$$I = \int_{y=0}^a \left[\log x \right]_0^b \cdot \frac{1}{y} \cdot dy$$

$$I = \int_0^a \log b \cdot \frac{1}{y} \cdot dy$$

$$I = \log_b [\log_a]_0^a$$

$$I = \log_a \cdot \log_b$$

$$4 \int_0^a \int_{x/a}^x \frac{x}{x^2 + y^2} dy \cdot dx$$

$$\text{Here } I = \int_0^a \int_{x/a}^x \frac{x}{x^2 + y^2} dy \cdot dx$$

$$I = \int_{x=0}^a \int_{y=x/a}^x \frac{x}{x^2 + y^2} dy \cdot dx$$

$$I = \int_{x=0}^a \frac{x}{x} \left[\tan^{-1} \left(\frac{y}{x} \right) \right]_{x/a}^x dx$$

$$I = \int_0^a \frac{x}{x} \left[\tan^{-1}(1) - \tan^{-1} \left(\frac{1}{a} \right) \right] dx$$

$$I = \int_0^a \frac{\pi}{4} \frac{x}{x} - \tan^{-1} \left(\frac{1}{a} \right) \frac{x}{x} \cdot dx$$

$$I = \int_0^a \left[\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{a}\right) \right] dx$$

$$= \left[\frac{\pi}{4} x - \tan^{-1}\left(\frac{1}{a}\right) x \right]_0^a$$

$$I = \frac{\pi}{4} a - \tan^{-1}\left(\frac{1}{a}\right) a$$

6 $\int_0^a \int_0^{\sqrt{ax-x^2}} x^2 + y^2 dx \cdot dy$

Here $I = \int_0^a \int_0^{\sqrt{ax-x^2}} x^2 + y^2 dx dy$

$$\therefore I = \int_{x=0}^a \int_{y=0}^{\sqrt{ax-x^2}} x^2 + y^2 dy dx$$

$$\therefore I = \int_0^a \left[x^2 y + \frac{y^3}{3} \right]_0^{\sqrt{ax-x^2}} dx$$

$$\therefore I = \int_0^a x^2 \sqrt{ax-x^2} + \frac{(\sqrt{ax-x^2})^3}{3} dx$$

$$\therefore I = \int_0^a x^2 \sqrt{ax-x^2} + \frac{\sqrt{ax-x^2}(ax-x^2)}{3} dx$$

સંપ અને શાંતિ માટે સહન કરવું ફરજિયાત છે.

$$I = \int_0^a x^2 \sqrt{ax-x^2} + \frac{ax \sqrt{ax-x^2}}{3} - \frac{x^2 \sqrt{ax-x^2}}{3} \cdot dx$$

$$I = \int_0^a \frac{2}{3} x^2 \sqrt{ax-x^2} + \frac{ax \sqrt{ax-x^2}}{3} \cdot dx$$

$$I = \int_{x=0}^a x^2 (ax-x^2)^{\frac{1}{2}} + \frac{(ax-x^2)^{\frac{3}{2}}}{3} \cdot dx$$

$$I = \int_0^a x^{5/2} (a-x)^{1/2} \cdot dx + \frac{1}{3} \int_0^a x^{3/2} (a-x)^{3/2} \cdot dx$$

Here, we take $x = at$

$$\therefore dx = a \cdot dt$$

x	a	1
t	0	0

$$\therefore I = \int_0^1 a^{5/2} + (a-at)^{1/2} + \frac{1}{3} \int_0^1 a^{3/2} + (a-at)^{3/2}$$

$$\therefore I = \int_0^1 a^4 + (1-t)^{1/2} \cdot dt + \frac{1}{3} \int_0^1 a^4 + (1-t)^{3/2} dt$$

તીવ્ર ઇચ્છાથી પચ્ચરની દિવાલને પણ તોડી શકાય છે.

$$\therefore I = a^4 \int_0^1 t^{\frac{7}{2}} (1-t)^{\frac{3}{2}} dt$$

$$+ \frac{a^4}{3} \int_0^1 t^{\frac{5}{2}} (1-t)^{\frac{5}{2}} dt$$

$$\therefore I = a^4 \frac{\sqrt{7/2} \cdot \sqrt{3/2}}{\sqrt{6}} + \frac{a^4}{3} \frac{\sqrt{5/2} \cdot \sqrt{5/2}}{\sqrt{5}}$$

$$\therefore I = \frac{a^4 6\pi^2}{4 \times 4 \times 4 \times 2}$$

$$\therefore I = \frac{3a^4 \pi^2}{64}$$

* Task 6: Change of variables from Cartesian to Polar Coordinates.

1 Evaluate $\iint_R \frac{4xy}{x^2+y^2} e^{-x^2-y^2} dx \cdot dy$ over

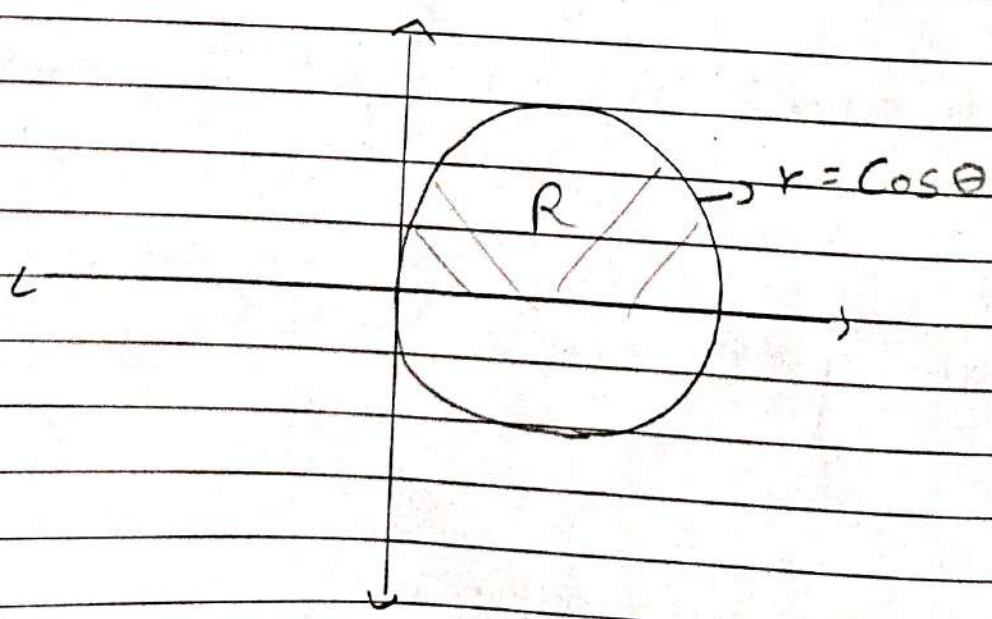
the region bounded by the circle,
 $x^2+y^2-x=0$ in first quadrant.

Here $x = r \cos \theta$ and $y = r \sin \theta$

Now $x^2+y^2-x=0 \Rightarrow r^2 \cos^2 \theta + r^2 \sin^2 \theta - r \cos \theta =$

$\Rightarrow r^2 = r \cos \theta$

$\Rightarrow r = \cos \theta$



$$\Rightarrow \iint \frac{4xy}{x^2+y^2} e^{-(x^2+y^2)} dx dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\pi/2 \cos \theta} \frac{4r \cos \theta r \sin \theta}{r^2} e^{-r^2} r dr d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{\pi/2 \cos \theta} 2r \cdot \sin 2\theta \cdot e^{-r^2} dr \cdot d\theta$$

$$= - \int_{\theta=0}^{\pi/2} \int_{r=0}^{\pi/2 \cos \theta} (-2r) \sin 2\theta e^{-r^2} dr \cdot d\theta$$

$$= - \int_0^{\pi/2} \sin 2\theta \left[e^{-r^2} \right]_0^{\pi/2 \cos \theta} \cdot d\theta$$

$$= - \int_0^{\pi/2} \sin 2\theta \left[e^{-\cos^2 \theta} - e^0 \right] d\theta$$

$$= - \left[e^{-\cos^2 \theta} + \frac{\cos 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{1}{e}$$

2 Evaluate $\int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy$

Here we take $x = r \cos \theta$, $y = r \sin \theta$

$$I = \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} \cdot r \cdot dr \cdot d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} \int_0^{\infty} (-2r) \cdot e^{-r^2} dr \cdot d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} [-e^{-r^2}]_0^{\infty} \cdot d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} (-1) \cdot d\theta$$

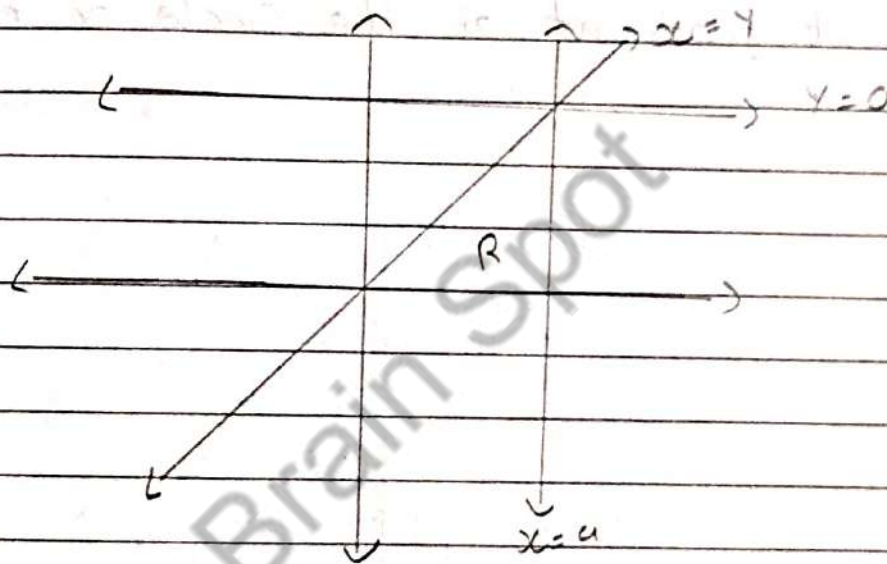
$$= \frac{1}{2} [\theta]_0^{\pi/2}$$

$$= \frac{\pi}{4}$$

3 Evaluate $\int_0^a \int_0^a \frac{x}{x^2 + y^2} dx dy$ by transforming

into polar coordinates.

$$I = \int_{y=0}^a \int_{x=y}^a \frac{x}{x^2 + y^2} dx \cdot dy$$



$$\frac{\pi}{4} \quad a/\cos\theta$$

$$I = \int_{\theta=0}^{\pi/4} \int_{r=0}^{a/\cos\theta} \frac{r \cos\theta}{r^2} \cdot r \cdot dr \cdot d\theta$$

$$I = \int_{\theta=0}^{\pi/4} [1]_0^{a/\cos\theta} \cdot \cos\theta \cdot d\theta$$

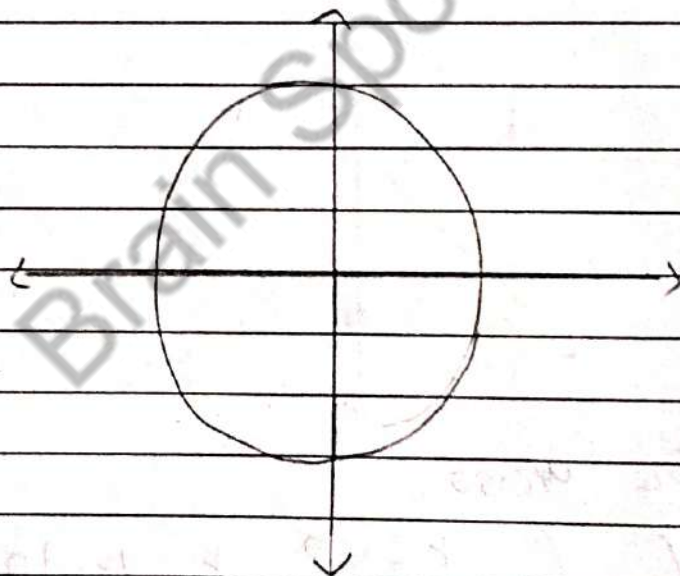
$$I = \int_{\theta=0}^{\pi/4} \frac{a}{\cos\theta} \cdot \cos\theta \cdot d\theta$$

$$I = \left[\theta \right]_0^{\pi/4} \cdot a$$

$$I = \frac{\pi}{4} a$$

4 Evaluate $\iint_R \frac{1-x^2-y^2}{1+x^2+y^2} dx dy$ over the

First quadrant of the circle $x^2 + y^2 = 1$



Here, $x = r \cos \theta$ and $y = r \sin \theta$

Hence, $x^2 + y^2 = 1$

$$\therefore r^2 \cos^2 \theta + r^2 \sin^2 \theta = 1$$

$$\therefore r = 1$$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^1 \int \frac{1-r^2}{1+r^2} r \cdot dr \cdot d\theta$$

$$\text{Let } r^2 = \cos 2t$$

$$\therefore 2r \cdot dr = -2 \sin 2t \cdot dt$$

$$\therefore r \cdot dr = -\sin 2t \cdot dt$$

$$r=0 \Rightarrow t = \pi/4$$

$$r=1 \Rightarrow t = 0$$

$$\therefore I = \int_{\theta=0}^{\pi/2} \int_{t=\pi/4}^0 \int \frac{1-\cos 2t}{1+\cos 2t} (-\sin 2t) dt \cdot d\theta$$

$$\therefore I = \int_0^{\pi/2} \int_{\pi/4}^0 \int \frac{2\sin^2 t}{2\cos^2 t} (-\sin 2t) dt \cdot d\theta$$

$$\therefore I = + \int_0^{\pi/2} \int_{\pi/4}^0 (1 - \cos 2t) dt \cdot d\theta$$

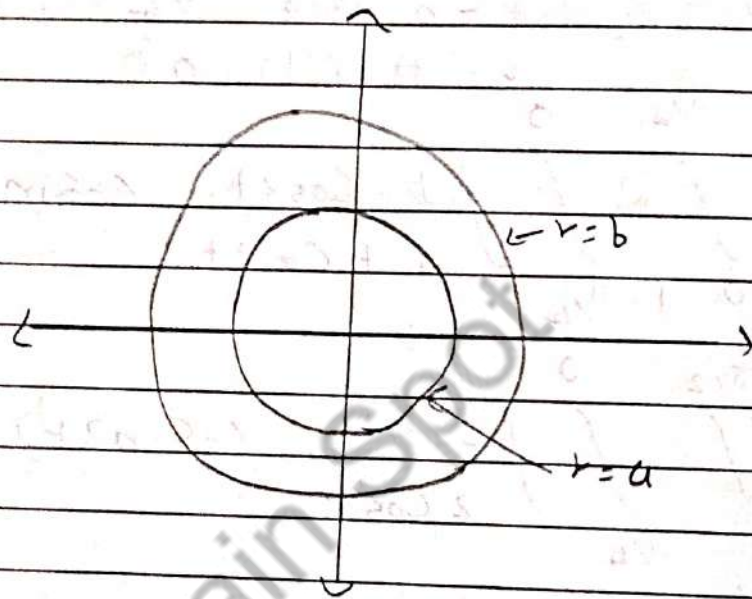
$$\therefore I = \int_0^{\pi/2} \left(t - \frac{\sin 2t}{2} \right) \Big|_{\pi/4}^0 d\theta$$

$$\therefore I = \int_0^{\pi/2} \left(\frac{\pi}{4} - \frac{1}{2} \right) d\theta$$

$$\therefore I = \frac{\pi}{8} (\pi - 2)$$

5 Evaluate $\iint_R \frac{x^2 y^2}{x^2 + y^2}$ over the region

bounded by the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$



$$I = \int_{\theta=0}^{2\pi} \int_{r=b}^a \frac{r^2 \cos^2 \theta \cdot r^2 \sin^2 \theta}{r^2 (\cos^2 \theta + \sin^2 \theta)} \cdot r \, dr \, d\theta$$

$$I = \int_{\theta=0}^{2\pi} \int_b^a r^4 \cos^2 \theta \sin^2 \theta \, dr \, d\theta$$

$$I = \int_{\theta=0}^{2\pi} \cos^2 \theta \sin^2 \theta \left[\frac{r^5}{5} \right]_b^a \, d\theta$$

$$I = \frac{1}{4} \int_0^{2\pi} 4 \cos^2 \theta \sin^2 \theta \left(\frac{a^4 - b^4}{4} \right) d\theta$$

$$I = \int_0^{2\pi} \frac{\sin^2 2\theta}{4} \cdot \frac{a^4 - b^4}{4} d\theta$$

$$I = \frac{a^4 - b^4}{16} \left[\frac{\sin 4\theta}{4} \right]_0^{2\pi}$$

$$I = \frac{a^4 - b^4}{16} \left[\frac{2\pi}{4} \right]$$

$$I = \frac{2\pi (a^4 - b^4)}{32} = \frac{\pi (a^4 - b^4)}{16}$$

* Task: 7: Change of Variables From Cartesian to other Coordinates

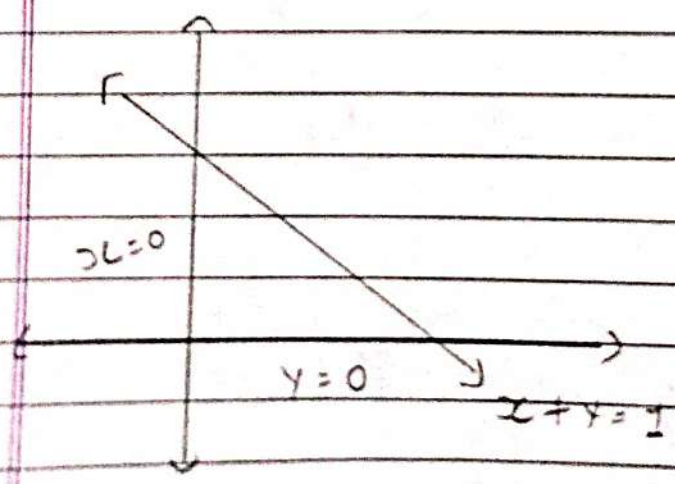
1 Using the transformation $x - y = u$, $x + y = v$, evaluate $\iint_R \cos \frac{x-y}{x+y} dx dy$

over the region bounded by the line $x = 0$, $y = 0$, $x + y = 1$

$\Rightarrow x - y = u, x + y = v$

$\therefore x = \frac{u+v}{2}, y = \frac{v-u}{2}$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -1$$

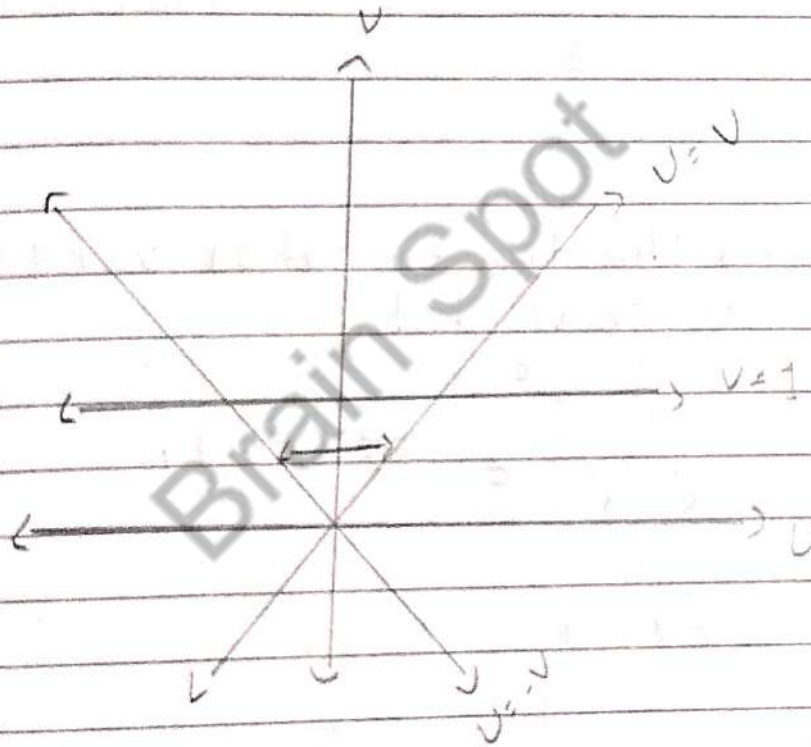


=> Given region,

$$x=0 \rightarrow 0 = \frac{u+v}{2} \Rightarrow u = -v$$

$$y=0 \rightarrow 0 = \frac{u-v}{2} \Rightarrow u = v$$

$$x+y=1 \rightarrow \frac{u+v}{2} + \frac{u-v}{2} = v = 1$$



$$I = \int_{v=0}^1 \int_{u=-v}^v \cos\left(\frac{v}{v}\right) \frac{1}{2} du \cdot dv$$

$$I = \frac{1}{2} \int_{v=0}^1 \left[\frac{\sin(u/v)}{1/v} \right]_{-v}^v \cdot dv$$

$$\begin{aligned} \therefore I &= \frac{1}{2} \int_{v=0}^1 v [\sin(v) - \sin(-v)] dv \\ &= \frac{1}{2} \sin(1) \int_0^1 2v dv \\ &= \frac{1}{2} \sin(1) \left[\frac{2v^2}{2} \right]_0^1 \\ &= \frac{\sin(1)}{2} \end{aligned}$$

2. Using the transformation $x + y = U$ and $y = UV$ show that,

$$\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}} dx dy$$

$\Rightarrow x + y = U, \quad y = UV$

$$I = \int_{x=0}^1 \int_{y=0}^{1-x} e^{\frac{y}{x+y}} dy dx$$

$\rightarrow x + UV = U$

$\therefore x = U(1 - V)$
 $y = UV$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -v \\ v & u \end{vmatrix} = u$$

$$\Rightarrow x=0 \rightarrow u(1-v)=0 \rightarrow u=0 \vee v=1$$

$$\Rightarrow x=1 \rightarrow u(1-v)=1 \rightarrow u=1 \vee v=0$$

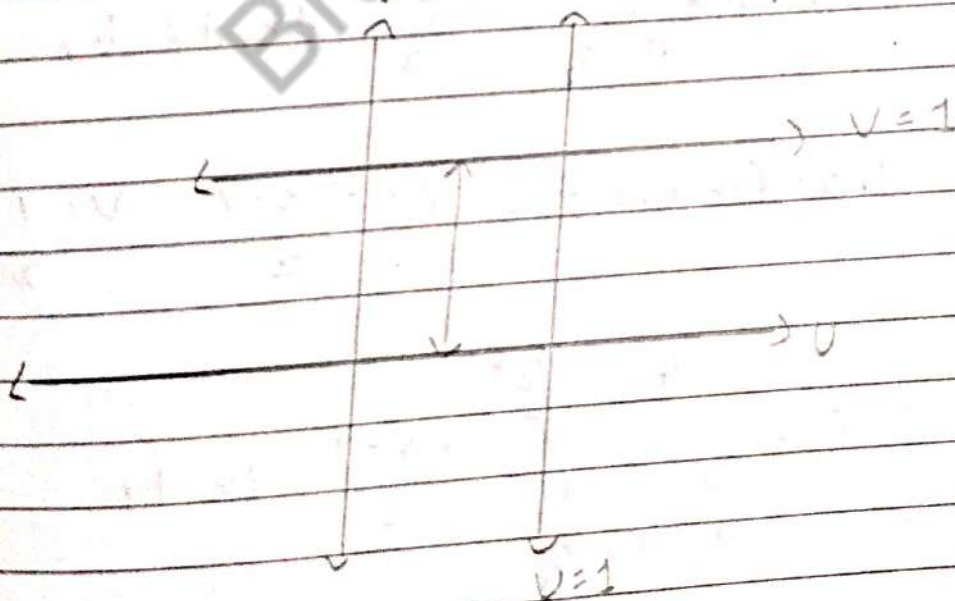
$$\Rightarrow y=0 \rightarrow 0=uv \Rightarrow u=0, v=0$$

$$\Rightarrow y=1-x \rightarrow x+y=1 \rightarrow u+uv+uv=1$$

$$\rightarrow u+2uv=1$$

$$\rightarrow u(1+2v)=1$$

$$\rightarrow u=1, v=0$$



$$I = \int_{v=0}^1 \int_{u=0}^1 e^v \cdot u \cdot du \cdot dv$$

$$I = \int_{v=0}^1 e^v \cdot \left[\frac{v^2}{2} \right]_0^1 dv$$

$$I = \int_{v=0}^1 \frac{e^v}{2} dv$$

$$I = \frac{1}{2} [e^v]_0^1$$

$$I = \frac{1}{2} [e^1 - e^0]$$

$$I = \frac{1}{2} [e - 1]$$

3 Evaluate $\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2} dx \cdot dy$ by

transformations $U = \frac{-2x-y}{2}$, $V = \frac{y}{2}$

$$I = \int_{y=0}^4 \int_{x=y/2}^{y/2+1} \frac{2x-y}{2} dx dy$$

$\Rightarrow 2U = -2x - y$

$\therefore 2U = -2x - 2V$

$$\Rightarrow U = \frac{x - y}{2}$$

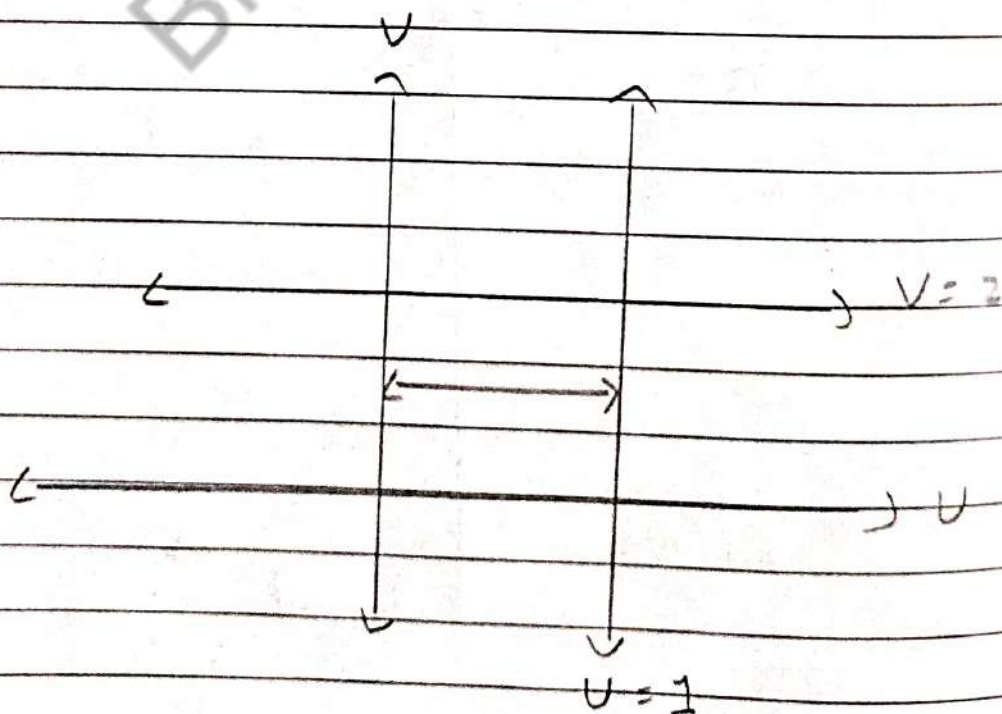
$$\rightarrow x = \frac{y}{2} \rightarrow U = \frac{y}{2} - \frac{y}{2} = 0$$

$$\rightarrow x = \frac{y}{2} + 1 \rightarrow U = \frac{y}{2} + 1 - \frac{y}{2} = 1$$

$$\Rightarrow V = \frac{y}{2}$$

$$\rightarrow y = 0 \rightarrow V = 0$$

$$\rightarrow y = 4 \rightarrow V = 2$$



$$I = \int_{v=0}^2 \int_{u=0}^1 U \cdot J \, du \, dv$$

$$\Rightarrow \bar{J}' = \frac{\partial(U, V)}{\partial(x, y)} = \frac{1}{2}$$

$$\Rightarrow I = \int_{v=0}^2 \frac{1}{2} \left[\frac{U^2}{2} \right]_0^1 dv$$

$$I = \int_0^2 \frac{1}{2} \cdot \frac{1}{4} dv$$

$$I = \left[\frac{v}{4} \right]_0^2$$

$$I = \frac{1}{2}$$

* Task 8 : Evaluation of triple integrals when limits given, Over the given region and Change of variable

1 Evaluate $\int_1^3 \int_1^{1/x} \int_0^{\sqrt{xy}} xy \, dz \, dy \, dx$

$$I = \int_{x=1}^3 \int_{y=1}^{1/x} \int_{z=0}^{\sqrt{xy}} xy \, dz \, dy \, dx$$

$$I = \int_{x=1}^3 \int_{y=1}^{1/x} [z]_0^{\sqrt{xy}} xy \, dy \, dx$$

$$I = \int_{x=1}^3 \int_{y=1}^{1/x} \sqrt{xy} \cdot xy \, dy \, dx$$

$$I = \int_{x=1}^3 \int_{y=1}^{1/x} x^{3/2} \cdot y \cdot \sqrt{y} \, dy \, dx$$

$$I = \int_{x=1}^3 x \sqrt{x} \left(\frac{2}{3} y^{3/2} \right)_{y=1}^{1/x} dx$$

$$I = \int_{x=1}^3 x \sqrt{x} \cdot \frac{2}{3} \left(\frac{1}{x^{3/2}} - 1 \right) dx$$

$$I = \frac{2}{3} \left(\log x - \frac{2}{3} x^{3/2} \right) \Big|_1^3$$

$$I = \frac{2}{3} \left[\log 3 - 2\sqrt{3} + \frac{2}{3} \right]$$

2 Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$

$$I = \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$$

$$I = \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \left[\frac{xy z^2}{2} \right]_0^{\sqrt{1-x^2-y^2}} dy \, dx$$

$$I = \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \frac{xy}{2} (1-x^2-y^2) dy \, dx$$

$$I = \frac{1}{2} \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \left(\frac{xy}{2} - \frac{x^3 y}{2} - \frac{y^3 x}{2} \right) dy \, dx$$

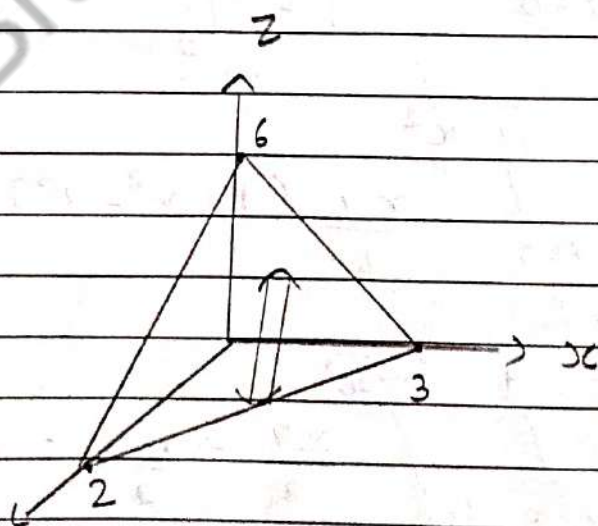
$$I = \frac{1}{2} \int_{x=0}^1 \left[\frac{x^2 y^2}{2} - \frac{x^4 y^4}{4} - \frac{y^4}{4} x^2 \right]_0^{\sqrt{1-x^2}} dx$$

$$\therefore I = \frac{1}{8} \int_0^1 x(1 - 2x^2 + x^4) dx$$

$$\therefore I = \frac{1}{8} \left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6} \right]_0^1$$

$$\therefore I = \frac{1}{48}$$

3 Evaluate $\iiint_V 2x \, dv$, where V is the region under the plane $2x + 3y + z = 6$ that lies in the first octant



$$I = \int_{x=0}^3 \int_{y=0}^{\frac{6-2x}{3}} \int_{z=0}^{6-2x-3y} 2x \, dz \cdot dy \cdot dx$$

$$I = 2 \int_{x=0}^3 \int_{y=0}^{\frac{6-2x}{3}} \int_0^{6-2x-3y} x \, dz \, dy \, dx$$

$$I = 2 \int_{x=0}^3 \int_0^{\frac{6-2x}{3}} (6-2x-3y) x \, dy \, dx$$

$$I = 2 \int_{x=0}^3 x \left(6-2x-3 \left[\frac{y^2}{2} \right]_0^{\frac{6-2x}{3}} \right) dx$$

$$I = 2 \int_{x=0}^3 x \left(6-2x - \frac{6-2x}{3} \right) dx$$

$$I = \frac{4}{3} \int_{x=0}^3 9x + x^3 - 6x \, dx$$

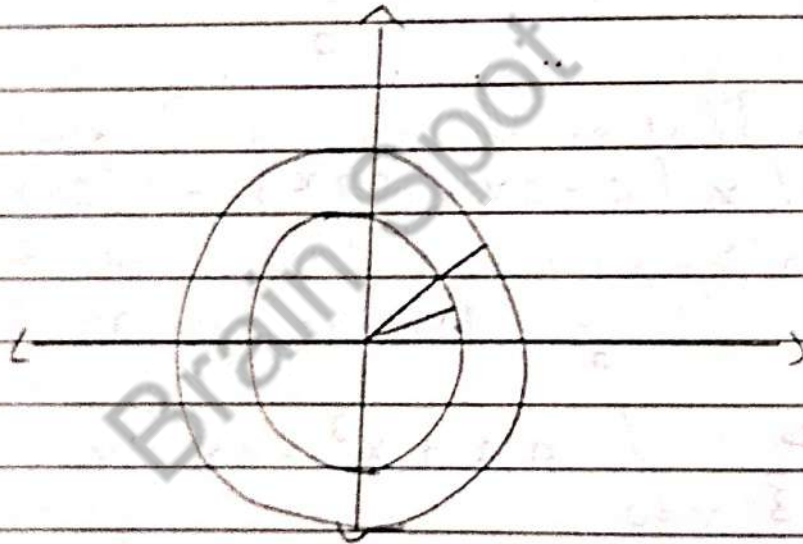
$$I = \frac{4}{3} \left[\frac{3x^2}{2} + \frac{x^4}{4} \right]_0^3$$

$$I = 9$$

4 Evaluate $\iiint_V \frac{1}{\sqrt{(x^2+y^2+z^2)^{3/2}}} dx dy dz$ over the

region bounded by the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, $a > b > 0$.

Here, $x = r \sin \theta \cos \phi$
 $y = r \sin \theta \sin \phi$
 $z = r \cos \theta$



$$\therefore I = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=a}^b \frac{r^2 \sin \theta}{r} dr d\theta d\phi$$

$$I = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[\frac{r^2}{2} \right]_a^b \sin \theta d\theta d\phi$$

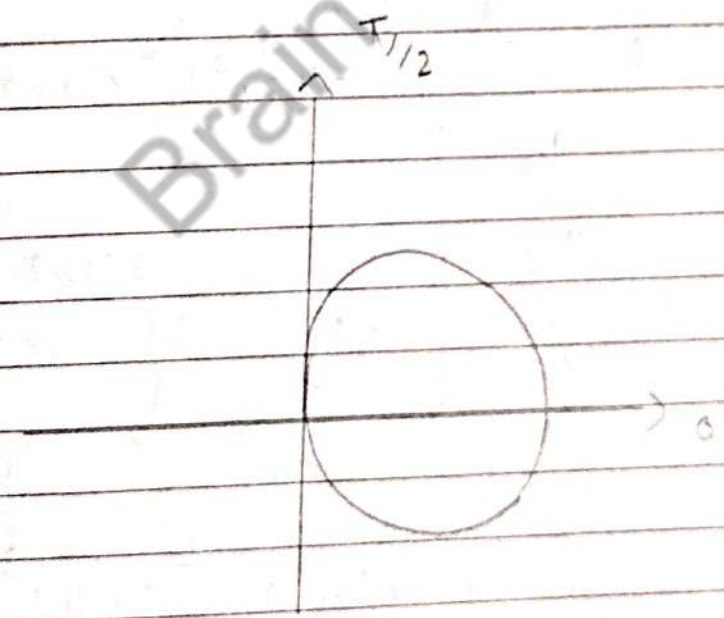
$$I = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{[a^2 - b^2]}{2} \sin \theta d\theta d\phi$$

$$I = \frac{a^2 - b^2}{2} \int_{\phi=0}^{2\pi} [\cos\theta]_0^{\pi} d\phi$$

$$I = \frac{a^2 - b^2}{2} \int_0^{2\pi} (\cos\pi - \cos 0) d\phi$$

$$I = \frac{4\pi (a^2 - b^2)}{2}$$

5 Evaluate $\iiint z^2 dx dy dz$ over the region $x^2 + y^2 + z^2 = 4$ by and the cylinder $x^2 + y^2 = 2x$



Here, $x = r \cos\theta$, $y = r \sin\theta$ and $z = z$

$$\Rightarrow \begin{aligned} x^2 + y^2 + z^2 &= 4 \\ r^2 + z^2 &= 4 \end{aligned}$$

$$z = \pm \sqrt{4 - r^2}$$

$$\Rightarrow x^2 + y^2 = 2x$$

$$\therefore r^2 = 2r \cos \theta$$

$$\therefore r = 2 \cos \theta$$

$$\rightarrow \theta = -\pi/2 + \pi/2$$

$$\therefore I = \int_{\theta = -\pi/2}^{\pi/2} \int_{r=0}^{2 \cos \theta} \int_{\phi = -\sqrt{4-r^2}}^{\sqrt{4-r^2}} 2 \cdot r \, dr \, d\theta \, d\phi$$

$$I = \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} \frac{-2}{3} (4 - r^2)^{3/2} \cdot r \, dr \, d\theta$$

$$I = \frac{-1}{3} \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} [-4 - r^2]^{3/2} (-2r) \, dr \, d\theta$$

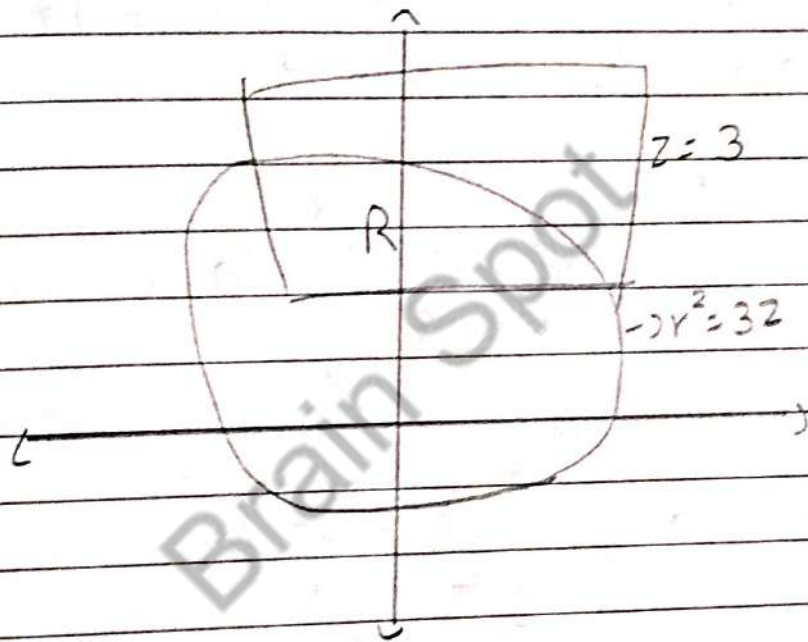
$$I = \frac{1}{3} \int_{-\pi/2}^{\pi/2} \left[\frac{2(4 - r^2)^{3/2}}{8} \right]_{r=0}^{2 \cos \theta} d\theta$$

$$I = \frac{1}{3} \int_{-\pi/2}^{\pi/2} [4 - 4 \cos^2 \theta]^{3/2} - (4)^{3/2} d\theta$$

$$I = \frac{-2}{15} \int_{-\pi/2}^{\pi/2} (2^5 \sin^5 \theta - 25) d\theta$$

$$I = \frac{2^6 \pi}{15}$$

6 Evaluate $\iiint x^2 + y^2 \, dx \, dy \, dz$ over the region bounded by paraboloid $x^2 + y^2 = 3z$ and the plane $z = 3$



Here, $x = r \cos \theta$, $y = r \sin \theta$ and $z = z$

$$\rightarrow x^2 + y^2 = 3z$$

$$\therefore r^2 = 3(3)$$

$$\therefore r = 3$$

$$z = \frac{r^2}{3} = 3$$

$$I = \int_{\theta=0}^{2\pi} \int_{\phi=0}^3 \int_{\frac{z}{3}}^3 r^2 \cdot r \, dz \, dr \, d\theta$$

$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^3 \left[\frac{z^3}{3} \right]_{\frac{z}{3}}^3 r^3 \, dr \, d\theta$$

$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^3 r^3 \left(3 - \frac{r^3}{3} \right) \, dr \, d\theta$$

$$I = \int_{\theta=0}^{2\pi} \left[\frac{3r^4}{4} - \frac{r^6}{18} \right]_0^3 \, d\theta$$

$$I = \left[\theta \right]_0^{2\pi} \left[\frac{3^5}{4} - \frac{3^6}{18} \right]$$

$$I = \frac{81\pi}{2}$$

7 Evaluate $\iiint x^2 y^2 z^2 dx dy dz$ over the region bounded by the surface $xy = 4$, $xy = 9$, $yz = 1$, $yz = 4$, $zx = 25$, $zx = 49$

Here, $xy = U$, $yz = V$, $xz = W$

$$J = \frac{\partial(U, V, W)}{\partial(x, y, z)} = \begin{vmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{vmatrix}$$

$$= 2xyz$$

$$\partial U \partial V \partial W = |J| \partial x \partial y \partial z$$

$$\therefore \partial x \partial y \partial z = \frac{1}{2xyz} \partial U \partial V \partial W$$

$$= \frac{1}{2\sqrt{UVW}} \partial U \partial V \partial W$$

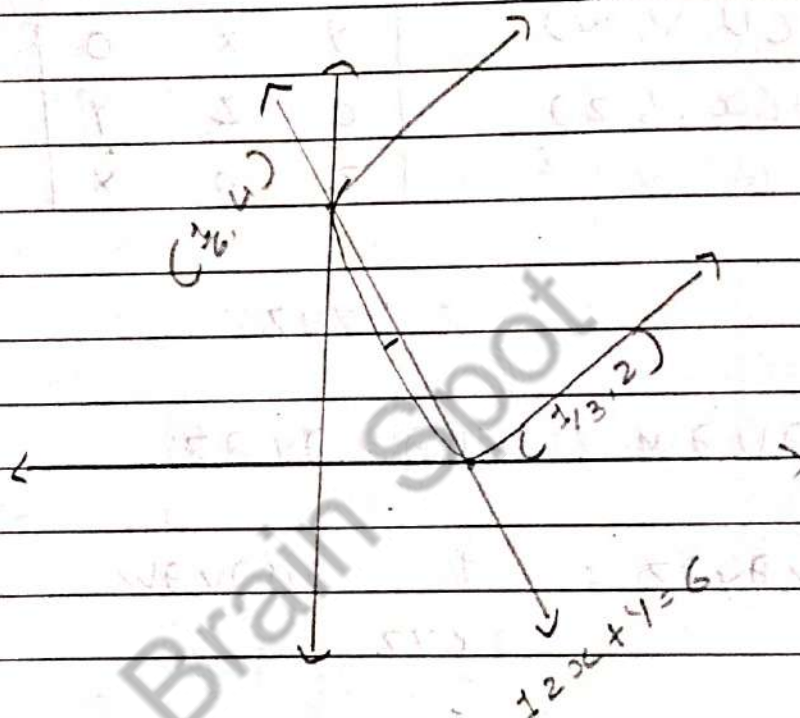
$$I = \int_{W=25}^{49} \int_{V=1}^4 \int_{U=4}^9 UVW \times \frac{1}{2\sqrt{UVW}} \partial U \partial V \partial W$$

$$I = \frac{4}{27} (343 - 125)(8 - 1)$$

$$I = \frac{115975}{27}$$

* Task 9: Area as Double Integral

1 Find the area between the rectangular hyperbola $3xy = 2$ and the line $12x + y = 6$



\Rightarrow Here, $12x + y = 6$

$$\therefore y = 6 - 12x$$

$$\therefore 3x(6 - 12x) = 2$$

$$\therefore x = \frac{1}{3}, \frac{1}{6}$$

$$\therefore y = 2, 4$$

$$\Rightarrow I = \int_{x=1/6}^{1/3} \int_{y=2/3x}^{6-12x} 1 \, dy \cdot dx$$

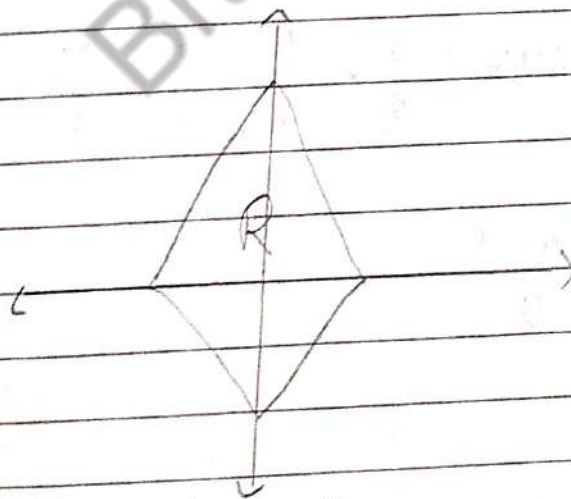
$$= \int_{1/6}^{1/3} \left(C - 12x - \frac{2}{3x} \right) dx$$

$$I = \left[Cx - \frac{12x^2}{2} - \frac{2}{3} \log x \right]_{1/6}^{1/3}$$

$$I = \frac{1}{2} - \frac{2}{3} \log 2$$

2 Find the area bounded by the hypocycloid

$$\left(\frac{x}{a} \right)^{2/3} + \left(\frac{y}{b} \right)^{2/3} = 1$$



$$\therefore \left(\frac{x}{a} \right)^{2/3} + \left(\frac{y}{b} \right)^{2/3} = 1$$

$$\therefore y = b \left[1 - \left(\frac{x}{a} \right)^{2/3} \right]^{3/2}$$

$$\therefore I = 4 \int_{x=0}^a b \left(1 - \left(\frac{x}{a}\right)^{2/3}\right)^{3/2} dx$$

Let $x = a \cos^3 t$

$$\therefore dx = 3a(\cos^2 t (-\sin t)) dt$$

$$x=0 \rightarrow t = \pi/2$$

$$x=a \rightarrow t = 0$$

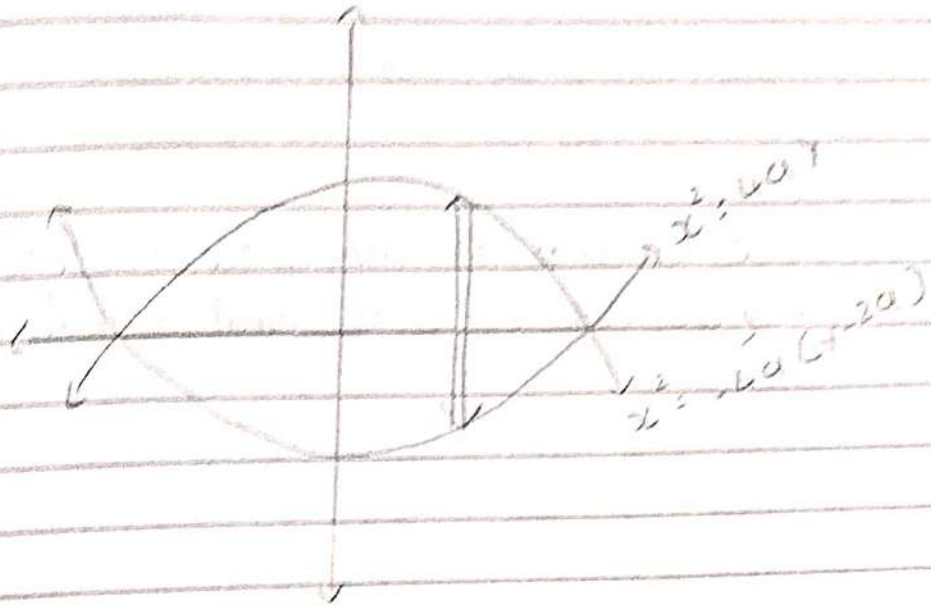
$$\therefore I = 4 \int_{\pi/2}^0 b (1 - \cos^2 t)^{3/2} (-3a)(\cos^2 t \sin t) dt$$

$$\therefore I = 12ab \int_0^{\pi/2} \sin^4 t \cdot \cos^2 t \cdot dt$$

$$\therefore I = 12ab \times \frac{3}{6} \times \frac{1}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$\therefore I = \frac{3ab\pi}{8}$$

3 Find the area bounded between the parabolas $x^2 = 4ay$ and $x^2 = -4a(y-2a)$



$$\therefore x^2 = 4ay$$

$$\therefore x^2 = -4a(y-2a)$$

$$\therefore 4ay = -4a(y-2a)$$

$$\therefore y = a$$

$$\therefore x^2 = \pm 2a$$

$$\therefore y^2 = \frac{x^2}{4a} = \frac{2a-x^2}{4a}$$

$$I = 2 \int_{x=0}^{2a} \int_{y=\frac{x^2}{4a}}^{\frac{2a-x^2}{4a}} 1 \cdot dy dx$$

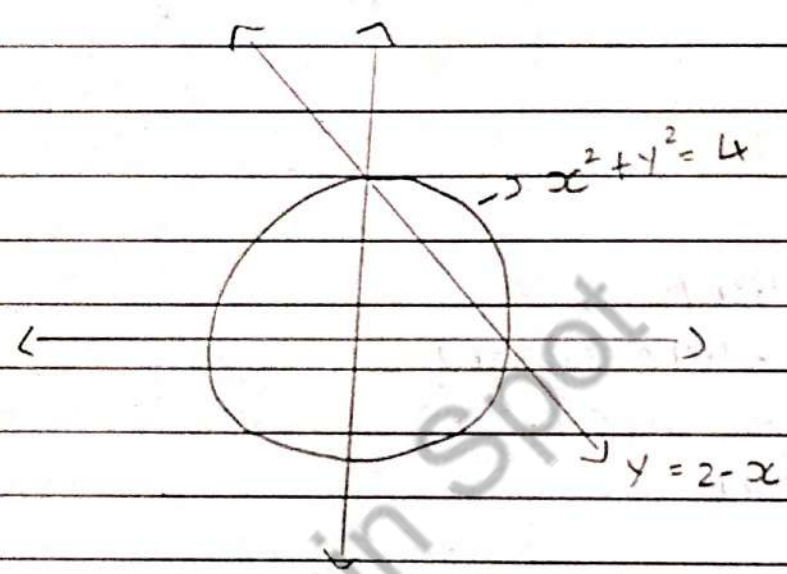
$$I = 2 \int_0^{2a} \left(\frac{2a-x^2}{4a} - \frac{x^2}{4a} \right) dx$$

$$I = 2 \left(\frac{2ax - \frac{x^3}{3}}{4a} \right)_0^{2a}$$

तीव्र धक्कावी पथरली दिवाले पर लोडी शकाय छे.

$$\therefore I = \frac{16a^2}{3}$$

4 Find smaller of the area enclosed by the curves $y = 2 - x$ and $x^2 + y^2 = 4$



$$\therefore x^2 + (2 - x)^2 = 4$$

$$\therefore x = 2 \text{ or } x = 0$$

$$\therefore y = 0 \text{ or } y = 2$$

$$I = \int_{x=0}^2 \int_{y=2-x}^{\sqrt{4-x^2}} 1 \, dy \cdot dx$$

$$\therefore I = \int_0^2 \sqrt{4-x^2} - 2 + x \, dx$$

$$\therefore I = \left[\frac{x}{2} \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} - 2x + \frac{x^2}{2} \right]_0^2$$

$$\therefore I = \pi - 2$$

5 Find the area between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=2\sin\theta}^{4\sin\theta} r \, dr \, d\theta$$

$$\therefore I = 2 \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_{2\sin\theta}^{4\sin\theta} d\theta$$

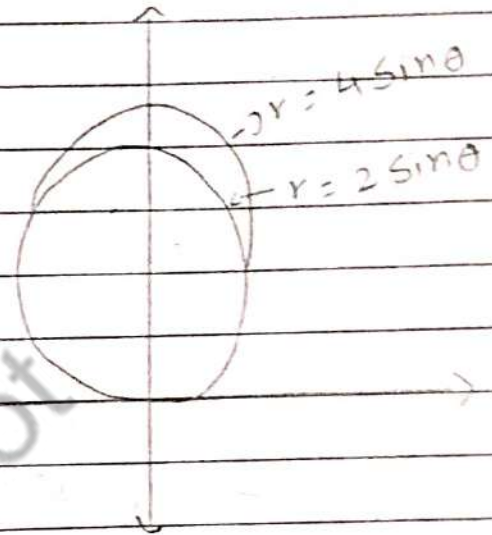
$$\therefore I = \int_0^{\pi/2} 16\sin^2\theta - 4\sin^2\theta \, d\theta$$

$$\therefore I = \int_0^{\pi/2} 12\sin^2\theta \, d\theta$$

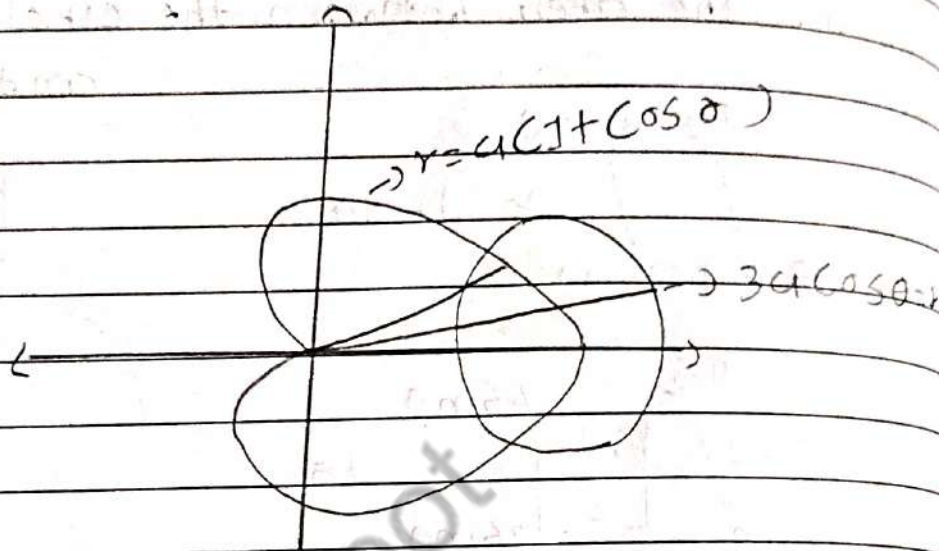
$$\therefore I = 6 \int_0^{\pi/2} 1 - \cos 2\theta \, d\theta$$

$$\therefore I = 6 \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$\therefore I = 3\pi$$



6 Find the area which lies inside the circle $r = 3a \cos \theta$ and $r = a(1 + \cos \theta)$



$$\therefore I = 2 \int_0^{\pi/3} \int_{r=a+a\cos\theta}^{3a\cos\theta} r \, dr \, d\theta$$

$$\therefore I = 2 \int_0^{\pi/3} \left[\frac{r^2}{2} \right]_{a(1+\cos\theta)}^{3a\cos\theta} d\theta$$

$$\therefore I = a^2 \int_0^{\pi/3} a^2 \cos^2 \theta - a^2 (1 + \cos \theta)^2 d\theta$$

$$\therefore I = a^2 \int_0^{\pi/3} 4(1 + \cos 2\theta - 1 - 2\cos \theta) d\theta$$

$$\therefore I = a^2 \left[\frac{3\theta + 4\sin 2\theta - 2\sin \theta}{2} \right]_0^{\pi/3}$$

$$I = a^2 \pi$$

3 Find the area common to cardioid
 $r = a(1 + \cos \theta)$ and $r = a(1 - \cos \theta)$

$$\therefore a(1 + \cos \theta) = a(1 - \cos \theta)$$

$$\therefore \cos \theta = 0$$

$$\therefore \theta = \frac{\pi}{2}$$

$$I = 4 \int_0^{\pi/2} \int_0^{a(1-\cos \theta)} r \, dr \, d\theta$$

$$I = 4 \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_0^{a(1-\cos \theta)} \cdot d\theta$$

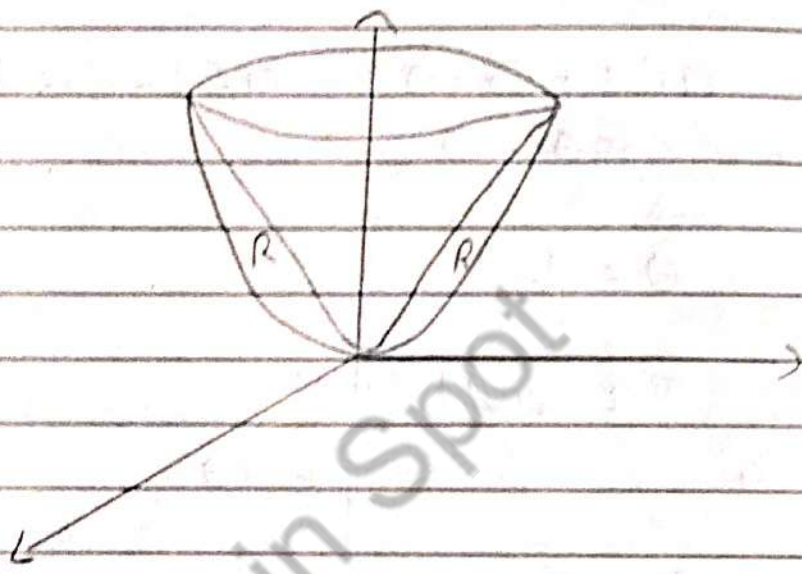
$$I = 2a^2 \int_0^{\pi/2} \left[1 - 2\cos \theta + \frac{1 + \cos 2\theta}{2} \right] d\theta$$

$$\therefore I = 2a^2 \left[\frac{3\theta}{2} - 2\sin \theta + \frac{\sin 2\theta}{4} \right]_0^{\pi/2}$$

$$\therefore I = 2a^2 \left(\frac{3\pi}{4} - 2 \right)$$

* Task 10: Volume as Triple Integral.

1 Find the Volume bounded by the cone $x^2 + y^2 = z^2$ and paraboloid $x^2 + y^2 = z$



$$\Rightarrow x^2 + y^2 = z^2, \quad x^2 + y^2 = z$$

$$\therefore r^2 = z^2, \quad \therefore r^2 = z$$

$$V = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^1 \int_{z=r^2}^r r \, dz \, dr \, d\theta$$

$$\therefore V = 4 \int_0^{\pi/2} \int_0^1 r \cdot (z)_{r^2}^r \, dr \, d\theta$$

$$\therefore V = 4 \int_0^{\pi/2} \int_0^1 r (r^2 - r^2) \, dr \, d\theta$$

$$\therefore V = 4 \int_0^{\pi/2} \left[\frac{r^3}{3} - \frac{r^4}{4} \right] d\theta$$

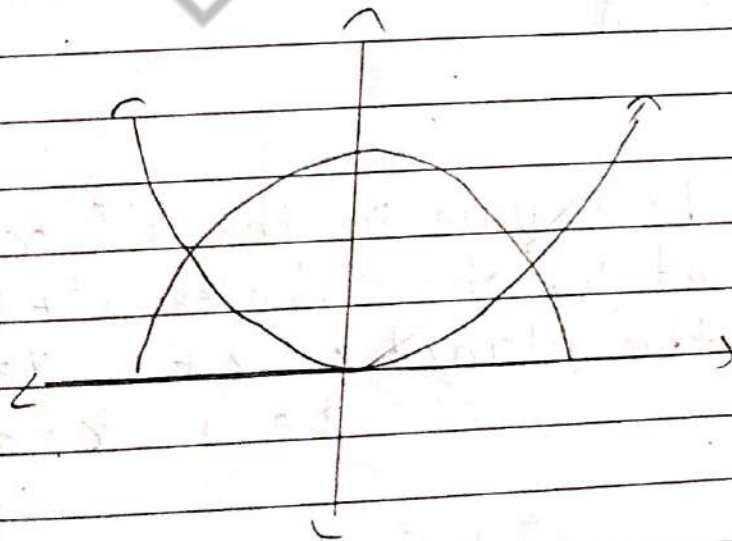
$$\therefore V = 4 \int_0^{\pi/2} \frac{1}{3} - \frac{1}{4} d\theta$$

$$\therefore V = 4 \left[\theta \right]_0^{\pi/2} \left(\frac{1}{3} - \frac{1}{4} \right)$$

$$\therefore V = 4 \cdot \frac{\pi}{2} \left(\frac{1}{3} - \frac{1}{4} \right)$$

$$\therefore V = \frac{\pi}{6}$$

2 Find the volume of the solid bounded by the paraboloid $x^2 + y^2 = z$ and $z = 4 - 3(x^2 + y^2)$



$$\Rightarrow x^2 + y^2 = z$$

$$z = 4 - 3r^2$$

$$\therefore z = r^2$$

$$\therefore z \rightarrow r^2 \rightarrow 4 - 3r^2$$

$$\therefore V = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^1 \int_{z=r^2}^{4-3r^2} r \, dz \, dr \, d\theta$$

$$\therefore V = 4 \int_0^{\pi/2} \int_0^1 r (4 - 3r^2 - r^2) \, dr \, d\theta$$

$$\therefore V = 4 \int_0^{\pi/2} \left[\frac{4r^2}{2} - \frac{4r^4}{4} \right]_0^1 d\theta$$

$$\therefore V = 4 \int_0^{\pi/2} 1 \, d\theta$$

$$\therefore V = 4 \left[\theta \right]_0^{\pi/2}$$

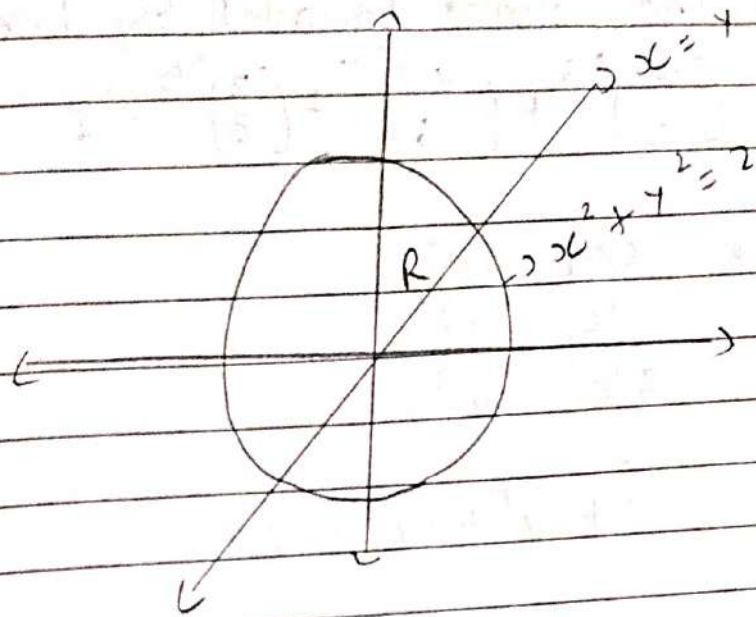
$$\therefore V = 2\pi$$

3 Find the volume in the 1st octant bounded by the cylinder $x^2 + y^2 = 2$ and the planes $z = x + y$, $y = x$, $z = 0$, $x = 0$

$$\therefore x^2 + y^2 = 2$$

$$\therefore 2x^2 = 2$$

$$\therefore x = \pm 1, \quad y = \pm 1$$



$$V = \int_{x=0}^1 \int_{y=x}^{\sqrt{2-x^2}} \int_{z=0}^1 1 \, dz \cdot dy \cdot dx$$

$$\therefore V = \int_0^1 \int_x^{\sqrt{2-x^2}} (x+y) \, dy \, dx$$

$$\therefore V = \int_0^1 \left[xy + \frac{y^2}{2} \right]_x^{\sqrt{2-x^2}} dx$$

$$\therefore V = \int_0^1 (x\sqrt{2-x^2} - x^2) + \frac{1}{2}(2-x^2-x^2) dx$$

$$\therefore V = \frac{2\sqrt{2}}{3}$$

4 Find the volume bounded by the solid

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1$$

Here, $x/a = U^3$
 $y/b = V^3$
 $z/c = W^3$

$\therefore U^2 + V^2 + W^2 = 1$

Here, $U = r \sin\theta \cos\phi$
 $V = r \sin\theta \sin\phi$
 $W = r \cos\theta$

$\Rightarrow J = \frac{\partial(x, y, z)}{\partial(U, V, W)} = \begin{vmatrix} 3ay^2 & 0 & 0 \\ 0 & 3bx^2 & 0 \\ 0 & 0 & 3cz^2 \end{vmatrix}$
 $= 27 a^3 v^2 u^2 w^2$

$\Rightarrow V = 8 \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} 27 a^3 v^2 u^2 w^2 du dv dw$

$V = 216 abc \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin^2\theta \cos^2\theta \cdot r^2 \sin^2\phi \sin^2\phi \cdot r^2 \cos^2\theta$
 $dr d\theta d\phi$

$\therefore V = \frac{4\pi}{35} abc$