

Unit-2 Partial Differentiation and its Applications

* Task 1: Limit of function of two variables.

$$1 \quad \lim_{(x, y) \rightarrow (0, 0)} \frac{4y^2x}{x^2 + y^2}$$

$$\rightarrow \lim_{x \rightarrow 0} \left\{ \lim_{y \rightarrow 0} \frac{4y^2x}{x^2 + y^2} \right\}$$

$$= \lim_{x \rightarrow 0} 0$$

$$= 0 \quad - \quad (1)$$

$$\rightarrow \lim_{y \rightarrow 0} \left\{ \lim_{x \rightarrow 0} \frac{4y^2x}{x^2 + y^2} \right\}$$

$$= \lim_{y \rightarrow 0} 0$$

$$= 0 \quad - \quad (2)$$

\rightarrow putting $y = mx$ taking $x \rightarrow 0$

$$= \lim_{x \rightarrow 0} \frac{4x(mx)^2}{x^2 + (mx)^2}$$

$$= \lim_{x \rightarrow 0} \frac{4mx^3}{x^2(1+m^2)}$$

$$= \lim_{x \rightarrow 0} \frac{4mx}{1+m^2}$$

$$= 0 \quad - \quad (3)$$

→ putting $y = mx^2$, taking $x \rightarrow 0$

$$= \lim_{x \rightarrow 0} \frac{4x(mx^2)^2}{x^2 + (mx^2)^2}$$

$$= \lim_{x \rightarrow 0} \frac{4x^5 m}{x^2 + mx^4}$$

$$= \lim_{x \rightarrow 0} \frac{4x^3 m}{1 + mx^2}$$

$$= 0 \quad - \quad (4)$$

From eqⁿ 1, 2, 3 and 4 values are equal
So $\lim_{(x,y) \rightarrow (0,0)} \frac{4y^2 x}{x^2 + y^2}$ exist.

$$2 \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$$

$$\rightarrow \lim_{x \rightarrow 0} \left\{ \lim_{y \rightarrow 0} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \right\}$$

$$= \lim_{x \rightarrow 0} \frac{x^2}{\sqrt{x}}$$

$$= 0 \quad - \quad (1)$$

$$\rightarrow \lim_{y \rightarrow 0} \left\{ \lim_{x \rightarrow 0} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \right\}$$

$$= \lim_{y \rightarrow 0} x \frac{1}{\sqrt{y}}$$

$$= 0 \quad - \quad (2)$$

\rightarrow putting $y = mx$, taking $x \rightarrow 0$

$$= \lim_{x \rightarrow 0} \frac{x^2 - x(mx)}{\sqrt{x} - \sqrt{mx}}$$

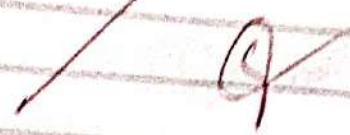
$$= 0 \quad - \quad (3)$$

\rightarrow putting $y = mx^2$ taking $x \rightarrow 0$

$$= \lim_{x \rightarrow 0} \frac{x^2 - x(mx^2)}{\sqrt{x} - \sqrt{mx^2}}$$

$= 0 - (1)$

From, eqⁿ 1, 2, 3 and 4 value are equal so, $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x - y}$ exist



3 $\lim_{(x,y) \rightarrow (2,1)} \frac{x^2 + 2y}{x + y^2}$

$\rightarrow \lim_{x \rightarrow 2} \left\{ \lim_{y \rightarrow 1} \frac{x^2 + 2y}{x + y^2} \right\}$

$= \lim_{x \rightarrow 2} \frac{x^2 + 2}{x + 1}$

$= \frac{4 + 2}{2 + 1}$

$= \frac{6}{3}$

$= 2 - (1)$

$\rightarrow \lim_{y \rightarrow 1} \left\{ \lim_{x \rightarrow 2} \frac{x^2 + 2y}{x + y^2} \right\}$

$$= \lim_{y \rightarrow 1} \frac{4+2y}{2+y^2}$$

$$= \frac{4+2}{2+1}$$

$$= \frac{6}{3}$$

$$= 2 \quad \text{--- (2)}$$

From eqⁿ 1 and 2 value are equal
So, $\lim_{(x,y) \rightarrow (1,2)} \frac{x^2+2y}{x+y^2}$ exist at 2

$$4 \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2}$$

$$\rightarrow \lim_{x \rightarrow 0} \left\{ \lim_{y \rightarrow 0} \frac{x^3 - y^3}{x^2 + y^2} \right\}$$

$$= \lim_{x \rightarrow 0} \frac{x^3}{x^2}$$

$$= 0 \quad \text{--- (1)}$$

$$\rightarrow \lim_{y \rightarrow 0} \left\{ \lim_{x \rightarrow 0} \frac{x^3 - y^3}{x^2 + y^2} \right\}$$

$$= \lim_{y \rightarrow 0} \frac{-y^3}{y^2}$$

$$= 0 - \textcircled{2}$$

-> putting $y = mx$ taking $x \rightarrow 0$

$$= \lim_{x \rightarrow 0} \frac{x^3 - m^3 x^3}{x^2 + m^2 x^2}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 (x - m^3 x)}{x^2 (1 + m^2)}$$

$$= \lim_{x \rightarrow 0} \frac{x - m^3 x}{1 + m^2}$$

$$= 0 - \textcircled{3}$$

-> putting $y = mx^2$ taking $x \rightarrow 0$

$$= \lim_{x \rightarrow 0} \frac{x^3 - m^3 x^4}{x^2 + m^2 x^3}$$

$$= \lim_{x \rightarrow 0} \frac{x^3 (x - m^3 x)}{x^2 (1 + mx)}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 - m^3 x}{1 + mx}$$

$$= 0 - \textcircled{4}$$

From eqⁿ 1, 2, 3 and 4 value are equal.

So, $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2}$ limit exist

$$5 \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2}$$

$$\rightarrow \lim_{x \rightarrow 0} \left\{ \lim_{y \rightarrow 0} \frac{x^2 y}{x^4 + y^2} \right\}$$
$$= 0 \quad - (1)$$

$$\rightarrow \lim_{x \rightarrow 0} \left\{ \lim_{x \rightarrow 0} \frac{x^2 y}{x^4 + y^2} \right\}$$
$$= 0 \quad - (2)$$

\rightarrow putting $y = mx$ taking $x \rightarrow 0$

$$= \lim_{x \rightarrow 0} \frac{x^2 (mx)}{x^4 + m^2 x^2}$$

$$= \lim_{x \rightarrow 0} \frac{x^3 m}{x^2 (x^2 + m^2)}$$

$$= \lim_{x \rightarrow 0} \frac{x m}{1 + m^2}$$

$= 0 - (3)$

-> putting $y = mx^2$ taking $x \rightarrow 0$

$$= \lim_{x \rightarrow 0} \frac{x^2(mx^2)}{x^4 + m^2x^3}$$

$$= \lim_{x \rightarrow 0} \frac{x^4 m}{x^3(x + m^2)}$$

$$= \lim_{x \rightarrow 0} \frac{xm}{x + m^2}$$

$= 0 -$

Brain Spot

Form eqⁿ 1, 2, 3 and 4 value are equal.

So, $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2}$ exist at 0.

$$6 \quad f(x, y) = \begin{cases} \frac{x}{\sqrt{x^2 + y^2}} & : \text{if } (x, y) \neq (0, 0) \\ 0 & : \text{if } (x, y) = (0, 0) \end{cases}$$

given $F(0, 0) = 0$

$$\rightarrow \lim_{x \rightarrow 0} \left\{ \lim_{y \rightarrow 0} \frac{x}{\sqrt{x^2 + y^2}} \right\}$$

$$\lim_{x \rightarrow 0} \frac{y}{x}$$

$$= 0 = \textcircled{1}$$

$$\Rightarrow \lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} \frac{x}{\sqrt{x^2 + y^2}} \right)$$

$$= 0 = \textcircled{2}$$

\Rightarrow putting $y = mx$ taking $x \rightarrow 0$

$$= \lim_{x \rightarrow 0} \frac{x}{x \sqrt{1 + m^2}}$$

$$= \lim_{x \rightarrow 0} \frac{1}{\sqrt{1 + m^2}}$$

$$= \frac{1}{\sqrt{1 + m^2}}$$

Here, last limit depends on m and m is not fix.

So, $\lim_{(x,y) \rightarrow (0,0)}$ is not exist at 0 .

* Task: 2 Continuity of function of two variables.

$$1 \quad f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} & ; \text{ if } (x, y) \neq (0, 0) \\ 0 & ; \text{ if } (x, y) = (0, 0) \end{cases}$$

$$\rightarrow \lim_{(x, y) \rightarrow (0, 0)} \frac{x^3 - y^3}{x^2 + y^2}$$

$$\rightarrow \lim_{x \rightarrow 0} \left\{ \lim_{y \rightarrow 0} \frac{x^3 - y^3}{x^2 + y^2} \right\}$$

$$= \lim_{x \rightarrow 0} \frac{x^3}{x^2}$$

$$= \lim_{x \rightarrow 0} x$$

$$= 0 \quad \text{--- (1)}$$

$$\rightarrow \lim_{y \rightarrow 0} \left\{ \lim_{x \rightarrow 0} \frac{x^3 - y^3}{x^2 + y^2} \right\}$$

$$= \lim_{y \rightarrow 0} \frac{-y^3}{y^2}$$

$$= \lim_{y \rightarrow 0} -y$$

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In this
to initialize

$$= 0 - \textcircled{2}$$

→ putting $y = mx$ taking $x \rightarrow 0$

$$= \lim_{x \rightarrow 0} \frac{x^3 - m^3 x^3}{x^2 + m^2 x^2}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 (x - m^3 x)}{x^2 (1 + m^2)}$$

$$= \lim_{x \rightarrow 0} \frac{x - m^3 x}{1 + m^2}$$

$$= 0 - \textcircled{3}$$

→ putting $y = mx^2$ taking $x \rightarrow 0$

$$= \lim_{x \rightarrow 0} \frac{x^3 - m^3 x^4}{x^2 + m^2 x^4}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 (x - x^2 m^3)}{x^2 (1 + m x^2)}$$

$$= \lim_{x \rightarrow 0} \frac{x - x^2 m^3}{1 + m x^2}$$

$$= 0 - \textcircled{4}$$

From eqⁿ 1, 2, 3 and 4 are equal
So $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2}$ exist.

hence $F(x, y)$ Continues at $x=0$.

$$2 \quad f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & ; \text{if } (x, y) \neq (0, 0) \\ 4 & ; \text{if } (x, y) = (0, 0) \end{cases}$$

given $f(0, 0) = 4$

$$\rightarrow \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 - y^2}{x^2 + y^2}$$

$$\rightarrow \lim_{x \rightarrow 0} \left\{ \lim_{y \rightarrow 0} \frac{x^2 - y^2}{x^2 + y^2} \right\}$$

$$= \lim_{x \rightarrow 0} \frac{x^2}{x^2}$$

$$= 1 - \textcircled{1}$$

$$\rightarrow \lim_{y \rightarrow 0} \left\{ \lim_{x \rightarrow 0} \frac{x^2 - y^2}{x^2 + y^2} \right\}$$

$$= \lim_{y \rightarrow 0} \frac{-y^2}{y^2}$$

$$= -1 - \textcircled{2}$$

$F(x, y)$ is discontinues at $(0, 0)$.

$$3 \quad F(x, y) = \begin{cases} \frac{x^2 + 2y}{x + y^2} & ; \text{if } (x, y) \neq (2, 1) \\ 2 & ; \text{if } (x, y) = (2, 1) \end{cases}$$

given, $F(2, 1) = 2$

$$\rightarrow \lim_{(x, y) \rightarrow (2, 1)} \frac{x^2 + 2y}{x + y^2}$$

$$\rightarrow \lim_{x \rightarrow 2} \left\{ \lim_{y \rightarrow 1} \frac{x^2 + 2y}{x + y^2} \right\}$$

$$= \lim_{x \rightarrow 2} \frac{x + 2}{x + 1}$$

$$= \frac{6}{3} = \textcircled{1}$$

$$\rightarrow \lim_{y \rightarrow 1} \left\{ \lim_{x \rightarrow 2} \frac{x^2 + 2y}{x + y^2} \right\}$$

$$= \lim_{y \rightarrow 1} \frac{4 + 2(y)}{2 + y^2}$$

$$= \frac{6}{3} = \textcircled{2}$$

$F(x, y)$ is continuous at $(2, 1)$
 $\lim_{(x, y) \rightarrow (2, 1)} F(x, y) = F(2, 1)$

4. F(x, y)

$$f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & ; \text{if } (x, y) \neq (0, 0) \\ 0 & ; \text{if } (x, y) = (0, 0) \end{cases}$$

given $f(0, 0) = 0$

$$\rightarrow \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 - y^2}{x^2 + y^2}$$

$$\rightarrow \lim_{x \rightarrow 0} \left\{ \lim_{y \rightarrow 0} \frac{x^2 - y^2}{x^2 + y^2} \right\}$$

$$= \lim_{x \rightarrow 0} \frac{x^2}{x^2}$$

$$= 1$$

$$\rightarrow \lim_{y \rightarrow 0} \left\{ \lim_{x \rightarrow 0} \frac{x^2 - y^2}{x^2 + y^2} \right\}$$

$$= \lim_{y \rightarrow 0} \frac{-y^2}{y^2}$$

$$= -1$$

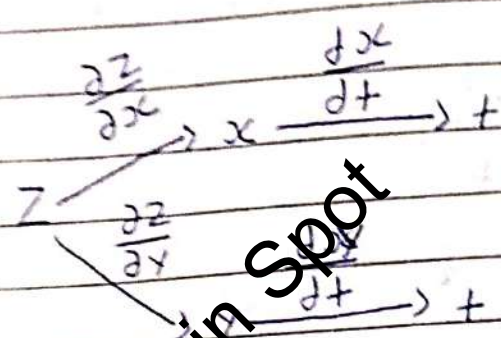
$f(x, y)$ is discontinuous at $(0, 0)$

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 - y^2}{x^2 + y^2} \neq f(0, 0)$$

* Task 6: Partial Differentiation of Composite Functions.

1. If $\tan^{-1}\left(\frac{x}{y}\right)$ and $x = 2t$ and $y = 1 - t^2$

then prove that $\frac{\partial z}{\partial t} = \frac{2}{1+t^2}$



Brain Spot

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{1}{1+x^2} \cdot 1 \cdot 2 + \frac{1}{1+x^2} \cdot \left(\frac{-x}{y^2}\right) (-2t)$$

$$= \frac{2y^2}{x^2+y^2} \cdot \frac{1}{y} + \frac{y^2}{x^2+y^2} \cdot \frac{2xt}{y^2}$$

$$= \frac{2y}{x^2+y^2} + \frac{2xt}{x^2+y^2}$$

$$= \frac{2(y+xt)}{x^2+y^2}$$

$$\frac{dz}{dt} = \frac{2(1-t^2 + 2t)}{4t^2 + 1 - 2t + t^4}$$

$$= \frac{2(1-t^2 + 2t^2)}{2t^2 + 1 + t^4}$$

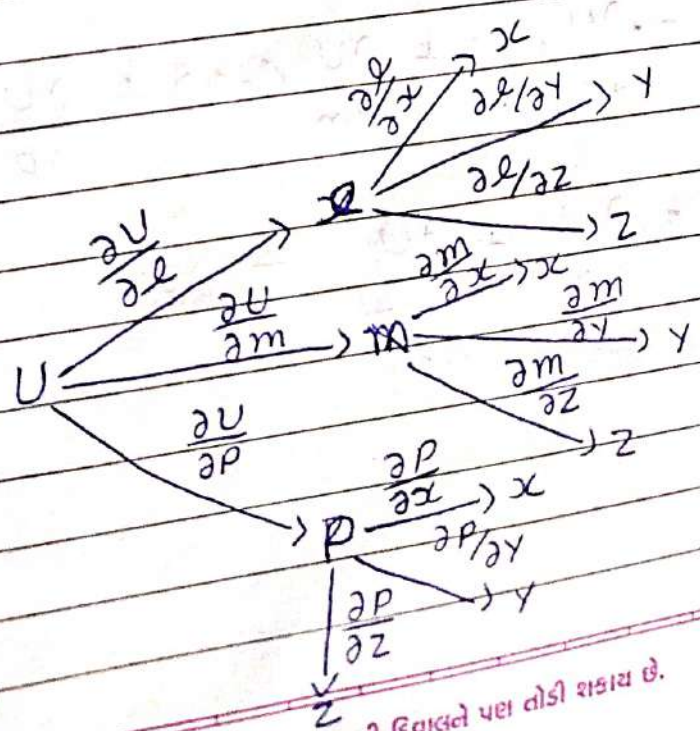
$$= \frac{2(1+t^2)}{(1+t^2)^2}$$

$$= \frac{2}{1+t^2}$$

Brain Spot

2 IF $U = F(x-y, y-z, z-x)$ then prove the

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0 \quad \begin{matrix} x-y = p \\ y-z = q \\ z-x = r \end{matrix}$$



$$\begin{aligned} \rightarrow \frac{\partial U}{\partial x} &= \frac{\partial U}{\partial r} \times \frac{\partial r}{\partial x} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial x} + \frac{\partial U}{\partial p} \times \frac{\partial p}{\partial x} \\ &= \frac{\partial U}{\partial r} (1) + \frac{\partial U}{\partial m} (0) + \frac{\partial U}{\partial p} (-1) \\ &= \frac{\partial U}{\partial r} - \frac{\partial U}{\partial p} \quad \text{--- (1)} \end{aligned}$$

$$\begin{aligned} \rightarrow \frac{\partial U}{\partial y} &= \frac{\partial U}{\partial r} \times \frac{\partial r}{\partial y} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial y} + \frac{\partial U}{\partial p} \times \frac{\partial p}{\partial y} \\ &= \frac{\partial U}{\partial r} (-1) + \frac{\partial U}{\partial m} (1) + \frac{\partial U}{\partial p} (0) \\ &= \frac{\partial U}{\partial m} - \frac{\partial U}{\partial r} \quad \text{--- (2)} \end{aligned}$$

$$\begin{aligned} \rightarrow \frac{\partial U}{\partial z} &= \frac{\partial U}{\partial r} \times \frac{\partial r}{\partial z} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial z} + \frac{\partial U}{\partial p} \times \frac{\partial p}{\partial z} \\ &= \frac{\partial U}{\partial r} (0) + \frac{\partial U}{\partial m} (-1) + \frac{\partial U}{\partial p} (1) \\ &= \frac{\partial U}{\partial p} - \frac{\partial U}{\partial m} \quad \text{--- (3)} \end{aligned}$$

$$eq^n \quad 1 + 2 + 3,$$

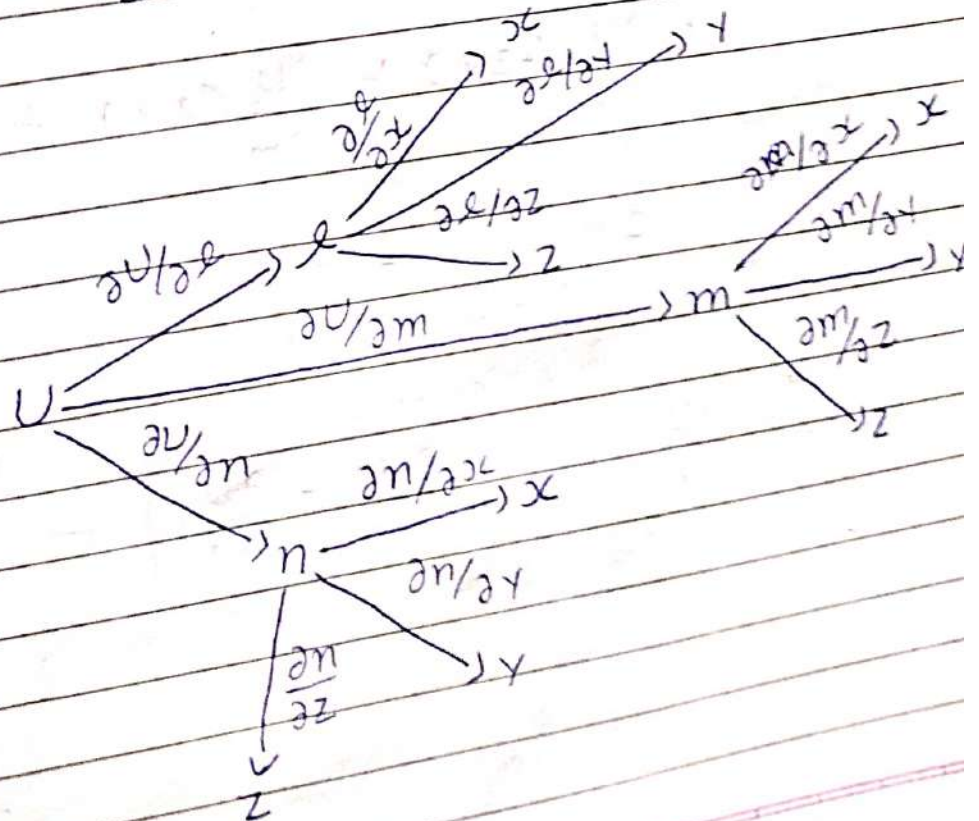
$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = \frac{\partial U}{\partial r} - \frac{\partial U}{\partial p} + \frac{\partial U}{\partial m} - \frac{\partial U}{\partial l} + \frac{\partial U}{\partial p} - \frac{\partial U}{\partial m}$$

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0$$

32. If $U = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$ prove that

$$\frac{1}{x} \frac{\partial U}{\partial x} + \frac{1}{y} \frac{\partial U}{\partial y} + \frac{1}{z} \frac{\partial U}{\partial z} = 0$$

We take, $x^2 - y^2 = l$
 $y^2 - z^2 = m$
 $z^2 - x^2 = n$



$$\rightarrow \frac{\partial U}{\partial x} = \frac{\partial U}{\partial z} \times \frac{\partial z}{\partial x} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial x} + \frac{\partial U}{\partial n} \times \frac{\partial n}{\partial x}$$

$$= \frac{\partial U}{\partial z} \times 2x + \frac{\partial U}{\partial m} (0) + \frac{\partial U}{\partial n} (-2x)$$

$$= \frac{\partial U}{\partial z} 2x - 2x \cdot \frac{\partial U}{\partial n}$$

$$= 2x \left(\frac{\partial U}{\partial z} - \frac{\partial U}{\partial n} \right)$$

$$\frac{1}{x} \left(\frac{\partial U}{\partial x} \right) = \left(\frac{\partial U}{\partial z} - \frac{\partial U}{\partial n} \right) \quad \text{--- (1)}$$

$$\rightarrow \frac{\partial U}{\partial y} = \frac{\partial U}{\partial z} \times \frac{\partial z}{\partial y} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial y} + \frac{\partial U}{\partial n} \times \frac{\partial n}{\partial y}$$

$$= \frac{\partial U}{\partial z} (-2y) + \frac{\partial U}{\partial m} (2y) + \frac{\partial U}{\partial n} (0)$$

$$= 2y \left(\frac{\partial U}{\partial m} - \frac{\partial U}{\partial z} \right)$$

$$\frac{1}{y} \left(\frac{\partial U}{\partial y} \right) = 2 \left(\frac{\partial U}{\partial m} - \frac{\partial U}{\partial z} \right) \quad \text{--- (2)}$$

$$\rightarrow \frac{\partial U}{\partial z} = \frac{\partial U}{\partial r} \times \frac{\partial r}{\partial z} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial z} + \frac{\partial U}{\partial n} \times \frac{\partial n}{\partial z}$$

$$= \frac{\partial U}{\partial r} (0) + \frac{\partial U}{\partial m} (-2z) + \frac{\partial U}{\partial n} (2z)$$

$$= 2z \left(\frac{\partial U}{\partial n} - \frac{\partial U}{\partial m} \right)$$

$$\frac{1}{z} \left(\frac{\partial U}{\partial z} \right) = 2 \left(\frac{\partial U}{\partial n} - \frac{\partial U}{\partial m} \right) \quad \text{--- (3)}$$

$$\text{eq}^1 + 2 + 3,$$

$$= \frac{1}{x} \left(\frac{\partial U}{\partial x} \right) + \frac{1}{y} \left(\frac{\partial U}{\partial y} \right) + \frac{1}{z} \left(\frac{\partial U}{\partial z} \right)$$

$$= 2 \left(\frac{\partial U}{\partial r} - \frac{\partial U}{\partial n} \right) + 2 \left(\frac{\partial U}{\partial m} - \frac{\partial U}{\partial r} \right) + 2 \left(\frac{\partial U}{\partial n} - \frac{\partial U}{\partial m} \right)$$

$$= 0$$

5 IF $U = F\left(\frac{y-x}{xy}, \frac{z-x}{zyc}\right)$ then prove

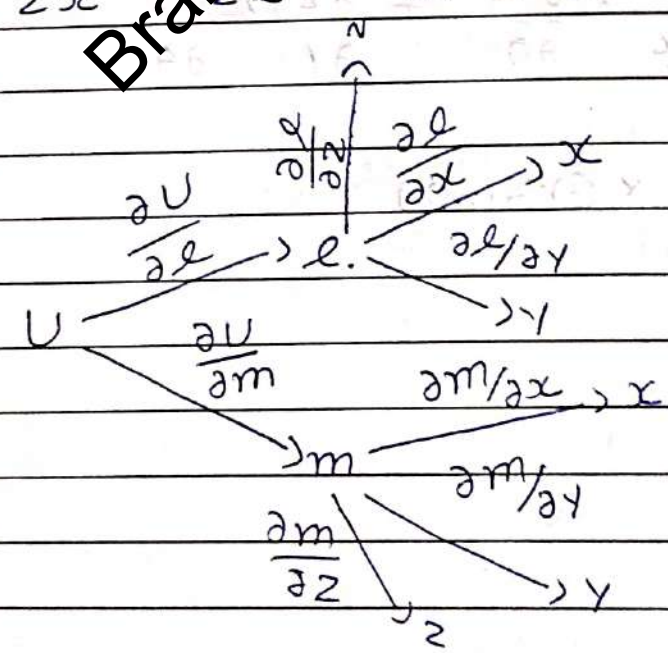
that, $x^2 \frac{\partial U}{\partial x} + y^2 \frac{\partial U}{\partial y} + z^2 \frac{\partial U}{\partial z} = 0$

Here we take $\frac{y-x}{xy} = l$, $\frac{z-x}{zyc} = m$

$\Rightarrow l = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$

$\Rightarrow m = \frac{z-x}{zyc} = \frac{1}{x} - \frac{1}{z}$

Brain Spot



$\Rightarrow \frac{\partial U}{\partial x} = \frac{\partial U}{\partial l} \times \frac{\partial l}{\partial x} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial x}$

$= \frac{\partial U}{\partial l} \left(-\frac{1}{x^2} \right) + \frac{\partial U}{\partial m} \left(-\frac{1}{x^2} \right)$

$$\Rightarrow x^2 \cdot \frac{\partial U}{\partial x} = -\frac{\partial U}{\partial r} - \frac{\partial U}{\partial m} \quad \text{--- (1)}$$

$$\Rightarrow \frac{\partial U}{\partial y} = \frac{\partial U}{\partial r} \times \frac{\partial r}{\partial y} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial y}$$

$$= \frac{\partial U}{\partial r} \left[+\frac{1}{y^2} \right] + \frac{\partial U}{\partial m} (0)$$

$$\Rightarrow \frac{\partial U \cdot y^2}{\partial y} = +\frac{\partial U}{\partial r} \quad \text{--- (2)}$$

$$\Rightarrow \frac{\partial U}{\partial z} = \frac{\partial U}{\partial r} \times \frac{\partial r}{\partial z} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial z}$$

$$= \frac{\partial U}{\partial r} (0) + \frac{\partial U}{\partial m} \left[+\frac{1}{z^2} \right]$$

$$\Rightarrow z^2 \cdot \frac{\partial U}{\partial z} = \frac{\partial U}{\partial m} \quad \text{--- (3)}$$

From eqⁿ 1 + 2 + 3,

$$\Rightarrow x^2 \cdot \frac{\partial U}{\partial x} + y^2 \cdot \frac{\partial U}{\partial y} + z^2 \cdot \frac{\partial U}{\partial z} = -\frac{\partial U}{\partial r} - \frac{\partial U}{\partial m} + \frac{\partial U}{\partial r}$$

$$+ \frac{\partial U}{\partial m}$$

$$\Rightarrow x^2 \cdot \frac{\partial U}{\partial x} + y^2 \cdot \frac{\partial U}{\partial y} + z^2 \cdot \frac{\partial U}{\partial z} = 0$$

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* Task 3: Partial Differentiation of function of two or more variables.

1 By using the definition find $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ and

$$\frac{\partial F}{\partial z}, \quad F(x, y, z) = x^2 y z^2 \quad \text{at point } (1, 2, 3)$$

$$\rightarrow \frac{\partial F}{\partial x} = 2x \cdot y \cdot z^2$$

$$\left(\frac{\partial F}{\partial x} \right)_{(1, 2, 3)} = 2(1) \cdot 2 \cdot (3)^2 = 36$$

$$\rightarrow \frac{\partial F}{\partial y} = x^2 z^2$$

$$\left(\frac{\partial F}{\partial y} \right)_{(1, 2, 3)} = (1)^2 \cdot (3)^2 = 9$$

$$\rightarrow \frac{\partial F}{\partial z} = x^2 y \cdot 2z$$

$$\left(\frac{\partial F}{\partial z} \right)_{(1, 2, 3)} = (1)^2 \cdot 2 \cdot 2(3) = 12$$

2 IF $U = e^{xyz}$ then Find $\frac{\partial^3 U}{\partial x \partial y \partial z}$

$$\rightarrow \frac{\partial U}{\partial z} = e^{xyz} \cdot xy$$

$$\rightarrow \frac{\partial}{\partial y} \left(\frac{\partial U}{\partial z} \right) = \frac{\partial^2 U}{\partial y \partial z} = e^{xyz} \cdot xz \cdot xy + e^{xyz} \cdot x$$

$$\rightarrow \frac{\partial}{\partial x} \left(\frac{\partial^2 U}{\partial y \partial z} \right) = e^{xyz} \cdot yz \cdot xy + e^{xyz} \cdot z \cdot xy + e^{xyz} \cdot xz \cdot y + e^{xyz} \cdot yz \cdot x + e^{xyz}$$

$$= e^{xyz} \cdot x^2 y^2 z^2 + e^{xyz} \cdot 3xyz + e^{xyz}$$

$$\frac{\partial^3 U}{\partial x \partial y \partial z} = e^{xyz} (1 + 3xyz + x^2 y^2 z^2)$$

3 IF $U = \tan(y + ax) + (y - ax)^{3/2}$ then show that,

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial y^2}$$

Given $U = \tan(y + ax) + (y - ax)^{3/2}$

$$\rightarrow \frac{\partial U}{\partial x} = \sec^2(y + ax) + \frac{3}{2} (y - ax)^{1/2} (-a)$$

$$\frac{\partial}{\partial x} \left(\frac{\partial U}{\partial x} \right) = \frac{\partial^2 U}{\partial x^2} = 2a \sec(y+ax) \cdot \sec(y+ax) \cdot \tan(y+ax) + \frac{3}{4} (y-ax)^{-\frac{1}{2}} \cdot a^2$$

$$\rightarrow \frac{\partial^2 U}{\partial x^2} = 2a \sec^2(y+ax) \cdot \tan(y+ax) + \frac{3}{4} (y-ax)^{-\frac{1}{2}} \cdot a^2 \quad \text{--- (1)}$$

$$\rightarrow \frac{\partial U}{\partial y} = \sec^2(y+ax) \cdot a + \frac{3}{2} (y-ax)^{\frac{1}{2}}$$

$$\rightarrow \frac{\partial^2 U}{\partial y^2} = 2 \sec(y+ax) \cdot \tan(y+ax) + \frac{3}{4} (y-ax)^{-1/2} \cdot a^2 \quad \text{--- (2)}$$

From eqⁿ 1 and 2

$$\therefore \frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial y^2} \text{ is equal.}$$

4 IF $U = e^{x^2 + y^2 + z^2}$ then prove that

$$\frac{\partial^3 U}{\partial x \partial y \partial z} = 8xyzU$$

$$\rightarrow \frac{\partial U}{\partial z} = e^{x^2 + y^2 + z^2} (2z)$$

$$\rightarrow \frac{\partial}{\partial y} \left(\frac{\partial U}{\partial z} \right) = \frac{\partial^2 U}{\partial y \partial z} = e^{x^2 + y^2 + z^2} (2z)(2y)$$

$$\frac{\partial}{\partial x} \left(\frac{\partial^2 U}{\partial y \partial z} \right) = \frac{\partial^3 U}{\partial x \partial y \partial z} = e^{x^2 + y^2 + z^2} (2z)(2y)(2x)$$

$$\frac{\partial^3 U}{\partial x \partial y \partial z} = 8xyzU$$

5 IF $U = x^2 \tan^{-1} \left(\frac{y}{x} \right) - y^2 \tan^{-1} \left(\frac{x}{y} \right)$ then

prove that $U_{xy} = U_{yx}$

$$\text{L.H.S.} = U_{xy} = \frac{\partial^2 U}{\partial x \partial y}$$

$$\frac{\partial U}{\partial y} = x^2 \cdot \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} + 0x \tan^{-1} \left(\frac{y}{x} \right)$$

$$-2y \cdot \tan^{-1}\left(\frac{x}{y}\right) + y^2 \cdot \frac{1}{1 + \frac{x^2}{y^2}} \quad \left(\frac{-y}{y^2}\right)$$

$$\frac{\partial U}{\partial y} = \frac{x^3}{x^2 + y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right) + \frac{xy^2}{x^2 + y^2}$$

$$= \frac{x(x^2 + y^2) - 2y \tan^{-1}\left(\frac{x}{y}\right)}{x^2 + y^2}$$

$$\rightarrow \frac{\partial U}{\partial y} = x - 2y \tan^{-1}\left(\frac{x}{y}\right)$$

$$\rightarrow \frac{\partial^2 U}{\partial x \partial y} = 1 - 2y \cdot \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{1}{y}$$

$$= 1 - \frac{2y^2}{x^2 + y^2}$$

$$= \frac{x^2 - y^2}{x^2 + y^2} \quad \text{--- (1)}$$

$$\text{R.H.S.} = U_{yx} = \frac{\partial^2 U}{\partial y \partial x}$$

$$\rightarrow \frac{\partial U}{\partial x} = 2x \cdot \tan^{-1}\left(\frac{y}{x}\right) + x^2 \cdot \frac{1}{1 + y^2} \cdot \left(\frac{-y}{x^2}\right)$$

$$- y^2 \cdot \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{1}{y}$$

$$\frac{\partial U}{\partial x} = 2x \cdot \tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2 y}{x^2 + y^2} - \frac{y^3}{x^2 + y^2}$$

$$= 2x \cdot \tan^{-1}\left(\frac{y}{x}\right) - y$$

$$\rightarrow \frac{\partial^2 U}{\partial y \partial x} = 2x \cdot \frac{1}{1 + y^2/x^2} \cdot 1/x - 1$$

$$= \frac{2x^2}{x^2 + y^2} - 1$$

$$= \frac{x^2}{x^2 + y^2} - 1 \quad \text{--- (2)}$$

\rightarrow By eqⁿ 1 and 2 L.H.S. = R.H.S.

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x}$$

* Task 4: Partial Differentiation of function of function.

1 IF $U = \log(x^3 + y^3 + z^3 - 3xyz)$ then prove that,

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 U = \frac{-9}{(x+y+z)^2}$$

$$\Rightarrow \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) U$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} \right)$$

Here, we take $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = V$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \cdot V \quad \text{--- (1)}$$

$$\Rightarrow V = \frac{[3x^2 - 3yz]}{x^3 + y^3 + z^3 - 3xyz} + \frac{[3y^2 - 3xz]}{x^3 + y^3 + z^3 - 3xyz}$$

$$+ \frac{[3z^2 - 3xy]}{x^3 + y^3 + z^3 - 3xyz}$$

$$V = \frac{3x^2 - 3yz + 3y^2 - 3xz + 3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\therefore V = \frac{-3(x^3 + y^3 + z^3 - yz - xy - xz)}{x^3 + y^3 + z^3 - 3xyz} \times \frac{(x+y+z)}{(x+y+z)}$$

$$\therefore V = \frac{3(x^3 + y^3 + z^3 - 3xyz)}{x^3 + y^3 + z^3 - 3xyz(x+y+z)}$$

$$\therefore V = \frac{3}{x+y+z}$$

put V value in eqⁿ 1

$$\therefore \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \cdot \frac{3}{x+y+z}$$

$$= \frac{\partial}{\partial x} \left(\frac{3}{x+y+z} \right) + \frac{\partial}{\partial y} \left(\frac{3}{x+y+z} \right) + \frac{\partial}{\partial z} \left(\frac{3}{x+y+z} \right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 U \text{ is prove.}$$

2 IF $U = \log(x^3 + y^3 - x^2y - xy^2)$ then prove that,

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 U = \frac{-4}{(x+y)^2}$$

$$\Rightarrow \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \cdot \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) U$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} \right)$$

Here we take $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = V$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) V \quad \text{--- (1)}$$

$$\Rightarrow V = \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y}$$

$$= \frac{(3x^2 - 2xy - y^2)}{x^3 + y^3 - x^2y - xy^2} + \frac{(3y^2 - x^2 - 2xy)}{x^3 + y^3 - x^2y - xy^2}$$

$$= \frac{2x^2 + 2y^2 - 2xy - 2xy}{x^3 + y^3 - x^2y - xy^2}$$

$$V = \frac{-2xy(-x+y) + 2y(y-x)}{-x^2(y-x) + y^2(y-x)}$$

$$\therefore V = \frac{(2y - 2x)(y - x)}{y^2 - x^2 (y - x)}$$

$$\therefore V = \frac{2(y - x)}{(y - x)(y + x)}$$

$$\therefore V = \frac{2}{y + x}$$

putting ~~v~~ value in eqⁿ 1

\therefore both side log,

$$\therefore \log V = 2 \log 2 - \log(y + x)$$

derivative w.r.t to x

$$\therefore \frac{1}{V} \left(\frac{\partial V}{\partial x} \right) = \frac{-1}{y+x} \quad \text{--- (2)}$$

derivative w.r.t to y

$$\therefore \frac{1}{V} \left(\frac{\partial V}{\partial y} \right) = \frac{1}{y+x} \quad \text{--- (3)}$$

adding eqⁿ 2 and 3

$$\therefore \frac{1}{V} \left(\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} \right) = \frac{-2}{y+x}$$

$$\therefore \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} = \frac{-2v}{(y+x)}$$

$$\therefore \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} = \frac{-2}{(y+x)^2}$$

$$\therefore \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 U = \frac{-4}{(x+y)^2} \text{ is prove}$$

3 IF $U = f(r)$, where $r^2 = x^2 + y^2$ then prove,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

$$\Rightarrow r^2 = x^2 + y^2$$

$U = f(r)$ is given,

$$\therefore \frac{\partial U}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\therefore \frac{\partial U}{\partial x} = f'(r) \cdot \frac{x}{r}$$

derivative w.r. to x

$$\therefore \frac{\partial^2 U}{\partial x^2} = f''(r) \frac{\partial r}{\partial x} + f'(r) \cdot \frac{1}{r} - f'(r) \frac{x}{r^2} \left(\frac{\partial r}{\partial x} \right)$$

$$\therefore \frac{\partial^2 U}{\partial x^2} = F''(r) \frac{x^2}{r^2} + \frac{F'(r)}{r} - F'(r) \frac{x^2}{r^3} \quad \text{--- (1)}$$

Similarly,

$$\therefore \frac{\partial^2 U}{\partial y^2} = F''(r) \frac{y^2}{r^2} + \frac{F'(r)}{r} - F'(r) \frac{y^2}{r^3} \quad \text{--- (2)}$$

adding eqⁿ 1 and 2,

$$\therefore \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = F''(r) \frac{(x^2 + y^2)}{r^2} + \frac{2F'(r)}{r}$$

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$$- F'(r) \frac{(x^2 + y^2)}{r^3}$$

$$= F''(r) \frac{(r^2)}{r^2} + \frac{2F'(r)}{r}$$

$$- F'(r) \frac{(r^2)}{r^3}$$

$$\therefore \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = F''(r) + \frac{1}{r} F'(r)$$

4 IF $U = f(r^2)$ where $r^2 = x^2 + y^2 + z^2$ then prove,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 4r^2 f''(r^2) + 6f'(r^2)$$

-> Here, $U = f(r^2)$

$$\therefore \frac{\partial U}{\partial x} = f'(r^2) \cdot 2r \cdot \frac{\partial r}{\partial x}$$

-> $r^2 = x^2 + y^2 + z^2$

$$\therefore 2r \cdot \frac{\partial r}{\partial x} = 2x$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

-> $\frac{\partial U}{\partial x} = f'(r^2) \cdot 2r \cdot \frac{x}{r}$

$$\frac{\partial U}{\partial x} = f'(r^2) \cdot 2x$$

∴ derivative w.r.t to x

$$\therefore \frac{\partial^2 U}{\partial x^2} = f''(r^2) \cdot 2r \cdot \frac{\partial r}{\partial x} \cdot 2x + 2f'(r^2)$$

$$\therefore \frac{\partial^2 U}{\partial x^2} = 4x^2 F''(r^2) + 2F'(r^2) \quad \text{--- (1)}$$

Similarly,

$$\frac{\partial^2 U}{\partial y^2} = 4y^2 (F''(r^2)) + 2F'(r^2) \quad \text{--- (2)}$$

$$\frac{\partial^2 U}{\partial z^2} = 4z^2 (F''(r^2)) + 2F'(r^2) \quad \text{--- (3)}$$

adding eqⁿ 1, 2 and 3

$$\begin{aligned} \therefore \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} &= 4F''(r^2)(x^2 + y^2 + z^2) \\ &\quad + 6F'(r^2) \\ &= 4F''(r^2)r^2 + 6F'(r^2) \end{aligned}$$

5 If $U = \log(\tan x + \tan y + \tan z)$ then prove that,

$$\sin(2x)U_x + \sin(2y)U_y + \sin(2z)U_z = 2$$

$$\Rightarrow U = \log(\tan x + \tan y + \tan z)$$

$$\frac{\partial U}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\rightarrow \sin(2x) \cdot \frac{\partial U}{\partial x} = \frac{2 \sin x \cdot \cos x}{\cos^2 x (\tan x + \tan y + \tan z)}$$

$$\sin(2x) \frac{\partial U}{\partial x} = \frac{2 \tan x}{\tan x + \tan y + \tan z} \quad - (1)$$

Similarly,

$$\rightarrow \sin(2y) \frac{\partial U}{\partial y} = \frac{2 \tan y}{\tan x + \tan y + \tan z} \quad - (2)$$

$$\rightarrow \sin(2z) \cdot \frac{\partial U}{\partial z} = \frac{2 \tan z}{\tan x + \tan y + \tan z} \quad - (3)$$

adding eq 1, 2 and 3

$$\therefore \sin 2x \cdot \frac{\partial U}{\partial x} + \sin 2y \cdot \frac{\partial U}{\partial y} + \sin 2z \left(\frac{\partial U}{\partial z} \right)$$

$$= \frac{2 (\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z}$$

$$= 2$$

6 If $U = (x^2 + y^2 + z^2)^{1/2}$ then prove that,

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = -U$$

$$\rightarrow U = (x^2 + y^2 + z^2)^{1/2}$$

$$\therefore \frac{\partial U}{\partial x} = \frac{-1}{2} \frac{(2x)}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\therefore \frac{\partial U}{\partial x} = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\rightarrow x \cdot \frac{\partial U}{\partial x} = \frac{-x^2}{(x^2 + y^2 + z^2)^{3/2}} \quad \text{--- (1)}$$

Similarly,

$$\rightarrow y \cdot \frac{\partial U}{\partial y} = \frac{-y^2}{(x^2 + y^2 + z^2)^{3/2}} \quad \text{--- (2)}$$

$$\rightarrow z \cdot \frac{\partial U}{\partial z} = \frac{-z^2}{(x^2 + y^2 + z^2)^{3/2}} \quad \text{--- (3)}$$

adding eqⁿ 1, 2 and 3,

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = - \frac{(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\therefore x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = -1$$

$(x^2 + y^2 + z^2)^{-1/2}$

$$\therefore x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = -U \text{ is prove.}$$

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* Task : 5 Partial Differentiation of Function in which some variable treated as constant.

1 IF $x = r \cos \theta$ and $y = r \sin \theta$ then Find,

- (a) $\left(\frac{\partial x}{\partial r}\right)_\theta$ (b) $\left(\frac{\partial y}{\partial \theta}\right)_r$ (c) $\left(\frac{\partial r}{\partial x}\right)_y$ (d) $\left(\frac{\partial \theta}{\partial y}\right)_x$

$\Rightarrow x^2 = r^2 \cos^2 \theta$
 $y^2 = r^2 \sin^2 \theta$

$\therefore x^2 + y^2 = r^2 \sin^2 \theta + r^2 \cos^2 \theta$
 $\therefore x^2 + y^2 = r^2$

a $\left(\frac{\partial x}{\partial r}\right)_\theta$

$\frac{\partial x}{\partial r} = \cos \theta$

b $\left(\frac{\partial y}{\partial \theta}\right)_r$

$\frac{\partial y}{\partial \theta} = r \cos \theta$

$$c \quad \left(\frac{\partial r}{\partial x} \right)_y$$

$$x^2 + y^2 = r^2$$

$$\therefore 2x = 2r \cdot \frac{\partial r}{\partial x}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$d \quad \left(\frac{\partial \theta}{\partial y} \right)_x$$

$$\therefore \frac{\sin \theta}{\cos \theta} = \frac{y}{x}$$

$$\therefore \tan \theta = \frac{y}{x}$$

$$\therefore \theta = \tan^{-1} \left(\frac{y}{x} \right)$$

$$\therefore \frac{\partial \theta}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \left(\frac{1}{x} \right)$$

$$= \frac{x}{x^2 + y^2}$$

2. IF $x = U \tan v$, $y = U \sec v$ then prove

$$\text{that, } \left(\frac{\partial U}{\partial x} \right)_y \left(\frac{\partial v}{\partial x} \right)_y = \left(\frac{\partial U}{\partial x} \right)_x \left(\frac{\partial v}{\partial y} \right)_x$$

We know that, $\sec^2 v - \tan^2 v = 1$

$$\therefore \frac{y^2}{U^2} - \frac{x^2}{U^2} = 1$$

$$\therefore y^2 - x^2 = U^2$$

$$\rightarrow \left(\frac{\partial U}{\partial x} \right)_y = -2x = U \left(\frac{\partial U}{\partial x} \right)_y$$

$$\left(\frac{\partial U}{\partial x} \right)_y = \frac{x}{U} \quad - (1)$$

$$\rightarrow x = U \tan v \quad \text{and} \quad y = \frac{U}{\cos v}$$

$$x = U \frac{\sin v}{\cos v} = \frac{U \cos v \cdot \sin v}{\cos v}$$

$$\therefore x = y \cdot \sin v$$

$$\therefore 1 = y \cos v \left(\frac{\partial v}{\partial x} \right)_y$$

$$\therefore \left(\frac{\partial v}{\partial x} \right)_y = \frac{1}{y \cos v} \quad - (2)$$

Here, $y^2 - x^2 = U^2$

$$\rightarrow \left(\frac{\partial U}{\partial y} \right)_x = 2y = 2U \left(\frac{\partial U}{\partial y} \right)$$

$$\therefore \left(\frac{\partial U}{\partial y} \right) = \frac{y}{U} \quad \text{--- (3)}$$

$$\rightarrow x = U \tan v \quad y = U \sec v$$

$$U = \frac{x}{\tan v} \quad y = \frac{x \cdot \cos v \cdot 1}{\sin v \cdot \cos v}$$

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$$\therefore y = \frac{x}{\sin v}$$

$$\therefore y \cdot \sin v = x$$

$$\therefore y \cdot \cos v \left(\frac{\partial U}{\partial y} \right)_x = x$$

$$\therefore \left(\frac{\partial U}{\partial y} \right)_x = \frac{x}{y \cos v} \quad \text{--- (4)}$$

$$\rightarrow \frac{x}{U} \cdot \frac{y}{y \cos v} = \frac{y \cdot x}{U \cdot y \cos v}$$

$$\therefore \frac{x}{U \cos v} = \frac{x}{U \cos v} \quad \text{--- (5)}$$

therefore,

$$\left(\frac{\partial U}{\partial x}\right)_y \left(\frac{\partial U}{\partial x}\right)_y = \left(\frac{\partial U}{\partial y}\right)_x \left(\frac{\partial U}{\partial y}\right)_x \text{ is prove.}$$

3 IF $U = ax + by$ and $V = bx - ay$ then Find.

$$\left(\frac{\partial U}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial U}\right)_V = \frac{a^2}{a^2 + b^2} \quad \text{and}$$

$$\left(\frac{\partial y}{\partial U}\right)_x \cdot \left(\frac{\partial U}{\partial y}\right)_V = \frac{a^2 + b^2}{a^2}$$

$$\rightarrow U = ax + by \quad \text{--- (1)}$$

$$\therefore \left(\frac{\partial U}{\partial x}\right)_y = a \quad \text{--- (2)}$$

$$\rightarrow V = bx - ay$$

$$\therefore ay = bx - V$$

$$\therefore y = \frac{bx - V}{a}$$

put y value in eqⁿ 1

$$\therefore U = ax + b \left(\frac{bx - V}{a} \right)$$

$$\therefore Ua = a^2 x + b^2 x - bV$$

derivative w.r.t. U

$$\therefore a = a^2 + b^2 \left(\frac{\partial x}{\partial U} \right)_V$$

$$\therefore \left(\frac{\partial x}{\partial U} \right)_V = \frac{a}{a^2 + b^2} \quad - (3)$$

-> We multiple eqⁿ and 3

$$\therefore \left(\frac{\partial U}{\partial x} \right)_V \cdot \left(\frac{\partial x}{\partial U} \right)_V = \frac{a^2}{a^2 + b^2}$$

-> $V = bx - ay$

$$\left(\frac{\partial V}{\partial y} \right)_x = -a \left(\frac{\partial y}{\partial V} \right)_x$$

$$\therefore \left(\frac{\partial y}{\partial V} \right)_x = \frac{1}{-a} \quad - (4)$$

-> $V = bx - ay$

$$\therefore bx = V + ay$$

$$\therefore x = \frac{V + ay}{b}$$

put x value in eqⁿ 1

$$\therefore U = a \left(\frac{v + ay}{b} \right) + by$$

$$\therefore Ub = av + a^2y + b^2y$$

derivative w.r.t to y ,

$$\therefore 0 = a \left(\frac{\partial v}{\partial y} \right)_U + a^2 + b^2$$

$$\therefore \left(\frac{\partial v}{\partial y} \right)_U = -\frac{a^2 + b^2}{a} \quad \text{--- (5)}$$

-> We multiply eqⁿ 4 and 5

$$\therefore \left(\frac{\partial y}{\partial v} \right)_x \cdot \left(\frac{\partial v}{\partial y} \right)_U = \frac{a^2 + b^2}{a^2}$$

4 If $x = r \cos \theta$ and $y = r \sin \theta$ then prove that,

$$\left[x \cdot \left(\frac{\partial x}{\partial r} \right)_\theta + y \cdot \left(\frac{\partial y}{\partial r} \right)_\theta \right]^2 = x^2 + y^2$$

$$\rightarrow x = r \cos \theta$$

$$\therefore \left(\frac{\partial x}{\partial r} \right)_\theta = \cos \theta$$

$$\rightarrow y = r \sin \theta$$

$$\therefore \left(\frac{\partial y}{\partial r} \right)_{\theta} = \sin \theta$$

$$\rightarrow \text{L.H.S.} = \left[x \cdot \left(\frac{\partial x}{\partial r} \right)_{\theta} + y \left(\frac{\partial y}{\partial r} \right)_{\theta} \right]^2$$

$$= \left[x (x \cos \theta) + y (\sin \theta) \right]^2$$

$$= \left[r \cos^2 \theta + r \sin^2 \theta \right]^2$$

$$= r^2$$

$$= x^2 + y^2$$

$$\text{L.H.S.} = \text{R.H.S.}$$

$$\left[x \left(\frac{\partial x}{\partial r} \right)_{\theta} + y \left(\frac{\partial y}{\partial r} \right)_{\theta} \right]^2 = x^2 + y^2$$

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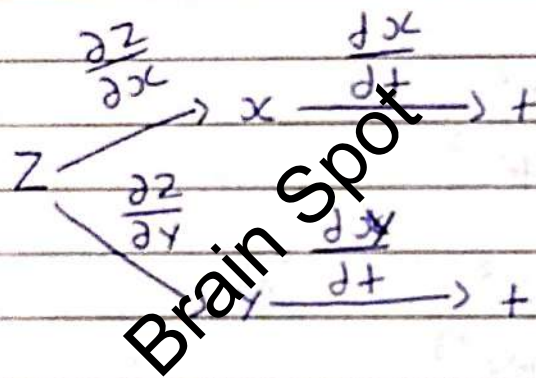
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* Task 6: Partial Differentiation of Composite Functions.

1. If $\tan^{-1}\left(\frac{x}{y}\right)$ and $x = 2t$ and $y = 1 - t^2$

then prove that $\frac{\partial z}{\partial t} = \frac{2}{1+t^2}$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{1}{y} \cdot 2 + \frac{1}{1 + \frac{x^2}{y^2}} \cdot \left(\frac{-x}{y^2}\right) (-2t)$$

$$= \frac{2y^2}{x^2 + y^2} \cdot \frac{1}{y} + \frac{y^2}{x^2 + y^2} \cdot \frac{2xt}{y^2}$$

$$= \frac{2y}{x^2 + y^2} + \frac{2xt}{x^2 + y^2}$$

$$= \frac{2(y + xt)}{x^2 + y^2}$$

$$\frac{dz}{dt} = \frac{2(1-t^2 + 2t)}{4t^2 + 1 - 2t + t^4}$$

$$= \frac{2(1-t^2 + 2t^2)}{2t^2 + 1 + t^4}$$

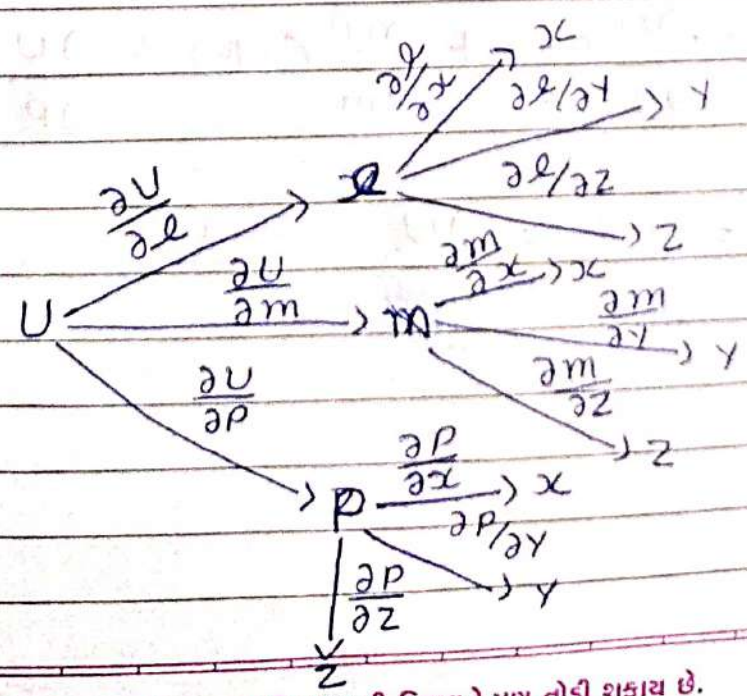
$$= \frac{2(1+t^2)}{(1+t^2)^2}$$

$$= \frac{2}{1+t^2}$$

Brain Spot

2 IF $U = F(x-y, y-z, z-x)$ then prove that

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0 \quad \begin{matrix} x-y = p \\ y-z = q \\ z-x = r \end{matrix}$$



$$\rightarrow \frac{\partial U}{\partial x} = \frac{\partial U}{\partial r} \times \frac{\partial r}{\partial x} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial x} + \frac{\partial U}{\partial p} \times \frac{\partial p}{\partial x}$$

$$= \frac{\partial U}{\partial r} (1) + \frac{\partial U}{\partial m} (0) + \frac{\partial U}{\partial p} (-1)$$

$$= \frac{\partial U}{\partial r} - \frac{\partial U}{\partial p} \quad \text{--- (1)}$$

$$\rightarrow \frac{\partial U}{\partial y} = \frac{\partial U}{\partial r} \times \frac{\partial r}{\partial y} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial y} + \frac{\partial U}{\partial p} \times \frac{\partial p}{\partial y}$$

$$= \frac{\partial U}{\partial r} (-1) + \frac{\partial U}{\partial m} (1) + \frac{\partial U}{\partial p} (0)$$

$$= \frac{\partial U}{\partial m} - \frac{\partial U}{\partial r} \quad \text{--- (2)}$$

$$\rightarrow \frac{\partial U}{\partial z} = \frac{\partial U}{\partial r} \times \frac{\partial r}{\partial z} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial z} + \frac{\partial U}{\partial p} \times \frac{\partial p}{\partial z}$$

$$= \frac{\partial U}{\partial r} (0) + \frac{\partial U}{\partial m} (-1) + \frac{\partial U}{\partial p} (1)$$

$$= \frac{\partial U}{\partial p} - \frac{\partial U}{\partial m} \quad \text{--- (3)}$$

eqⁿ 1 + 2 + 3,

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = \frac{\partial U}{\partial l} \frac{\partial l}{\partial x} + \frac{\partial U}{\partial m} \frac{\partial m}{\partial x} + \frac{\partial U}{\partial n} \frac{\partial n}{\partial x} + \frac{\partial U}{\partial l} \frac{\partial l}{\partial y} + \frac{\partial U}{\partial m} \frac{\partial m}{\partial y} + \frac{\partial U}{\partial n} \frac{\partial n}{\partial y} + \frac{\partial U}{\partial l} \frac{\partial l}{\partial z} + \frac{\partial U}{\partial m} \frac{\partial m}{\partial z} + \frac{\partial U}{\partial n} \frac{\partial n}{\partial z}$$

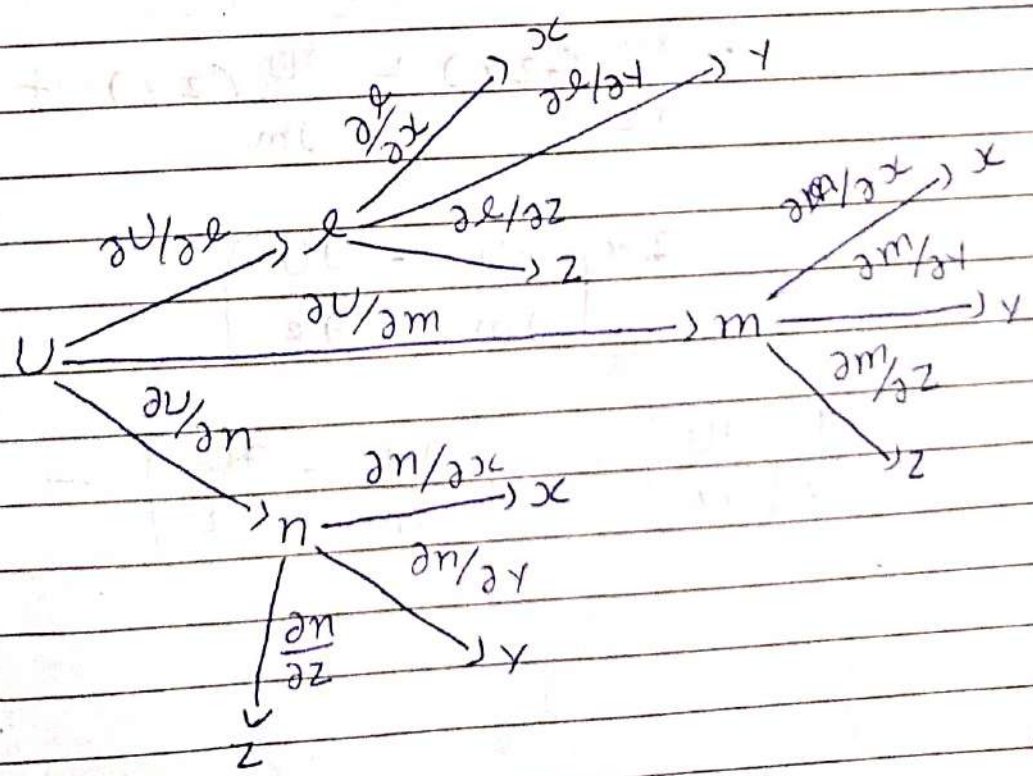
$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0$$

32 If $U = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$ prove that

$$\frac{1}{x} \frac{\partial U}{\partial x} + \frac{1}{y} \frac{\partial U}{\partial y} + \frac{1}{z} \frac{\partial U}{\partial z} = 0$$

Brain Spot

We take,
 $x^2 - y^2 = l$
 $y^2 - z^2 = m$
 $z^2 - x^2 = n$



$$\rightarrow \frac{\partial U}{\partial x} = \frac{\partial U}{\partial r} \times \frac{\partial r}{\partial x} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial x} + \frac{\partial U}{\partial n} \times \frac{\partial n}{\partial x}$$

$$= \frac{\partial U}{\partial r} \times 2x + \frac{\partial U}{\partial m} (0) + \frac{\partial U}{\partial n} (-2x)$$

$$= \frac{\partial U}{\partial r} 2x - 2x \cdot \frac{\partial U}{\partial n}$$

$$= 2x \left(\frac{\partial U}{\partial r} - \frac{\partial U}{\partial n} \right)$$

$$\frac{1}{x} \left(\frac{\partial U}{\partial x} \right) = 2 \left(\frac{\partial U}{\partial r} - \frac{\partial U}{\partial n} \right) \quad \text{--- (1)}$$

$$\rightarrow \frac{\partial U}{\partial y} = \frac{\partial U}{\partial r} \times \frac{\partial r}{\partial y} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial y} + \frac{\partial U}{\partial n} \times \frac{\partial n}{\partial y}$$

$$= \frac{\partial U}{\partial r} (-2y) + \frac{\partial U}{\partial m} (2y) + \frac{\partial U}{\partial n} (0)$$

$$= 2y \left(\frac{\partial U}{\partial m} - \frac{\partial U}{\partial r} \right)$$

$$\frac{1}{y} \left(\frac{\partial U}{\partial y} \right) = 2 \left(\frac{\partial U}{\partial m} - \frac{\partial U}{\partial r} \right) \quad \text{--- (2)}$$

$$\rightarrow \frac{\partial U}{\partial z} = \frac{\partial U}{\partial r} \times \frac{\partial r}{\partial z} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial z} + \frac{\partial U}{\partial n} \times \frac{\partial n}{\partial z}$$

$$= \frac{\partial U}{\partial r} (0) + \frac{\partial U}{\partial m} (-2z) + \frac{\partial U}{\partial n} (2z)$$

$$= 2z \left(\frac{\partial U}{\partial n} - \frac{\partial U}{\partial m} \right)$$

$$\frac{1}{2} \left(\frac{\partial U}{\partial z} \right) = 2 \left(\frac{\partial U}{\partial n} - \frac{\partial U}{\partial m} \right) \quad \text{--- (3)}$$

$$\text{eq}^1 \quad 1 \quad 2 + 3,$$

$$= \frac{1}{x} \left(\frac{\partial U}{\partial x} \right) + \frac{1}{y} \left(\frac{\partial U}{\partial y} \right) + \frac{1}{z} \left(\frac{\partial U}{\partial z} \right)$$

$$= 2 \left(\frac{\partial U}{\partial r} - \frac{\partial U}{\partial n} \right) + 2 \left(\frac{\partial U}{\partial m} - \frac{\partial U}{\partial r} \right) + 2 \left(\frac{\partial U}{\partial n} - \frac{\partial U}{\partial m} \right)$$

$$= 0$$

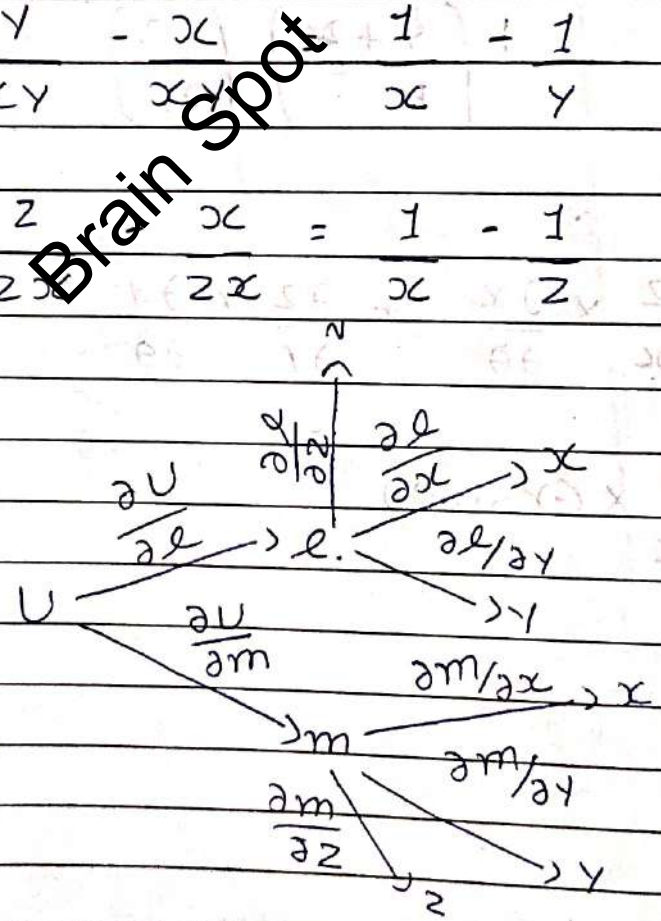
5 IF $U = F\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$ then prove

that, $x^2 \frac{\partial U}{\partial x} + y^2 \frac{\partial U}{\partial y} + z^2 \frac{\partial U}{\partial z} = 0$

Here we take $\frac{y-x}{xy} = l$, $\frac{z-x}{zx} = m$

$\Rightarrow l = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$

$\Rightarrow m = \frac{z-x}{zx} = \frac{1}{x} - \frac{1}{z}$



$\Rightarrow \frac{\partial U}{\partial x} = \frac{\partial U}{\partial l} \times \frac{\partial l}{\partial x} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial x}$

$= \frac{\partial U}{\partial l} \left(\frac{-1}{x^2} \right) + \frac{\partial U}{\partial m} \left(\frac{-1}{x^2} \right)$

$$\Rightarrow x^2 \cdot \frac{\partial U}{\partial x} = -\frac{\partial U}{\partial r} - \frac{\partial U}{\partial m} \quad \text{--- (1)}$$

$$\Rightarrow \frac{\partial U}{\partial y} = \frac{\partial U}{\partial r} \times \frac{\partial r}{\partial y} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial y}$$

$$= \frac{\partial U}{\partial r} \left[\frac{+1}{y^2} \right] + \frac{\partial U}{\partial m} (0)$$

$$\Rightarrow \frac{\partial U}{\partial y} \cdot y^2 = \frac{\partial U}{\partial r} \quad \text{--- (2)}$$

$$\Rightarrow \frac{\partial U}{\partial z} = \frac{\partial U}{\partial r} \times \frac{\partial r}{\partial z} + \frac{\partial U}{\partial m} \times \frac{\partial m}{\partial z}$$

$$= \frac{\partial U}{\partial r} (0) + \frac{\partial U}{\partial m} \left[\frac{+1}{z^2} \right]$$

$$\Rightarrow z^2 \cdot \frac{\partial U}{\partial z} = \frac{\partial U}{\partial m} \quad \text{--- (3)}$$

From eqⁿ 1 + 2 + 3,

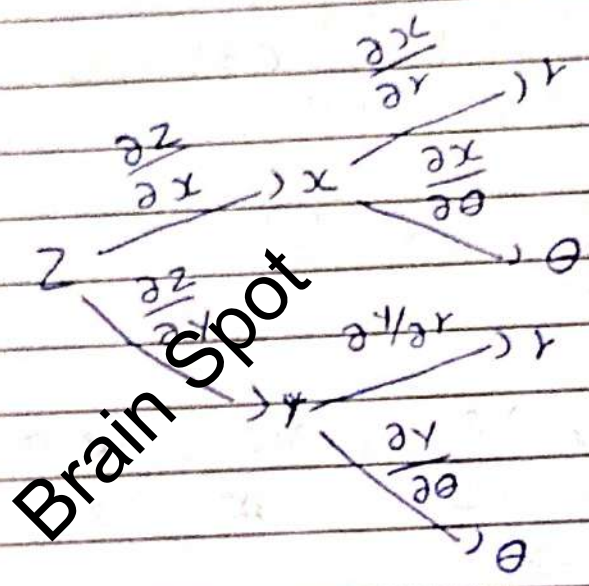
$$\Rightarrow x^2 \cdot \frac{\partial U}{\partial x} + y^2 \cdot \frac{\partial U}{\partial y} + z^2 \cdot \frac{\partial U}{\partial z} = -\frac{\partial U}{\partial r} - \frac{\partial U}{\partial m} + \frac{\partial U}{\partial r}$$

$$+ \frac{\partial U}{\partial m}$$

$$\Rightarrow x^2 \cdot \frac{\partial U}{\partial x} + y^2 \cdot \frac{\partial U}{\partial y} + z^2 \cdot \frac{\partial U}{\partial z} = 0$$

4 IF $z = f(x, y)$ where $x = r \cos \theta$, $y = r \sin \theta$ then show that,

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$



$$\Rightarrow \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta \quad \text{--- (1)}$$

$$\Rightarrow \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial \theta}$$

$$= -r \sin \theta \cdot \frac{\partial z}{\partial x} + r \cos \theta \cdot \frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{1}{r \cdot \theta} \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cos \theta - \sin \theta \cdot \frac{\partial z}{\partial y} \quad \text{--- (2)}$$

$$\begin{aligned} \text{L.H.S.} &= \left(\frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2 \\ &= \left(\frac{\partial z}{\partial x} \right)^2 \cos^2 \theta + \sin^2 \theta \cdot \left(\frac{\partial z}{\partial y} \right)^2 + 2 \sin \theta \cos \theta \cdot \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \end{aligned}$$

$$\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial y} \right)^2 \cos^2 \theta + \left(\frac{\partial z}{\partial x} \right)^2 \sin^2 \theta$$

$$- 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \sin \theta \cdot \cos \theta$$

$$= \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2$$

$$\text{L.H.S.} = \text{R.H.S.}$$

* Task 7: Euler's Theorem for Homogeneous Function

1 Check Function is Homogeneous or not

(i) $U = x^3 \sin\left(\frac{x}{y}\right)$

Replacing $x = xt$ and y by yt

$$\therefore F(xt, yt) = xt^3 \sin\left(\frac{xt}{yt}\right)$$

$$= t^3 \sin\left(\frac{x}{y}\right)$$

$$= t^3 (U)$$

U is a homogeneous function of degree $n=3$.

(ii) $U = \frac{x+y}{x^3+y^3}$

Replacing x by xt and y by yt

$$\therefore F(xt, yt) = \frac{xt + yt}{x^3t^3 + y^3t^3}$$

$$= t \frac{(x+y)}{t^3(x^3+y^3)}$$

$$F(xt, yt) = t^{-2}(U)$$

U is a homoge. function of degree
 $n = -2$.

ciii) $U = x^2y^3 + x^3y^2 + 7$

Replasing x by xt and y by yt

$$F(xt, yt) = x^{2+2}(y^{3+3}) + x^{3+3}(y^{2+2}) + 7$$

U is not homoge. function.

civ) $U = \log x - \log y$

Replasing x by xt and y by yt .

$$F(xt, yt) = \log(xt) - \log(yt)$$

$$= \log\left(\frac{xt}{yt}\right)$$

$$= \log\left(\frac{x}{y}\right)$$

$$= t^0(U)$$

U is a homoge. function of degree
 $n = 0$

$$(C.V) \quad 4 \sqrt{\frac{x^5 - y^5}{x - y}}$$

Replacing x by $x+t$ and y by $y+t$

$$F(x+t, y+t) = 4 \sqrt{\frac{x^5 + 5 - y^5 + 5}{x+t - y+t}}$$

$$= + \left(4 \sqrt{\frac{x^5 - y^5}{x - y}} \right)$$

$$= + C.V$$

U is a homogeneous function of degree $n=1$.

2 IF $U = \left(\frac{x^2 + y^2}{x + y} \right)$ then prove that

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = U$$

$$\text{Let } F(U) = \frac{x^2 + y^2}{x + y}$$

Replacing x by $x+t$ and y by $y+t$

$$F(x+t, y+t) = \frac{x^2 + t^2 + y^2 + t^2}{x+t + y+t}$$

$$= + \left[\frac{x^2 + y^2}{x + y} \right]$$

$$= + (U)$$

U is a homoge. function of degree $n=1$.

By Euler's theorem,

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = nU$$

$$= 1U$$

that prove, $x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = U$

3 IF $U = e^{x^2} F(x, y)$, prove that

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 2U \cdot \log U$$

$$\therefore \log U = x^2 F(x, y) \cdot \log e$$

Replacing x by $x+$ and y by $y+$

$$\therefore \log U = (x^2 + y^2) F(x+, y+) \cdot \log e$$

$$\therefore F(x+, y+) = +^2 (x^2 F(x, y)) \cdot \log e$$

$$= t^2 U$$

U is a homoge. function of degree
 $n = 2 \log U$

By Euler's theorem,

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = n \cdot U$$

$$= 2U \cdot \log U$$

Hence prove that

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 2U \cdot \log U$$

4 If $U = \sin^{-1}(x^2 + y^2)^{1/5}$ then prove that,

$$x^2 \cdot \frac{\partial^2 U}{\partial x^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 U}{\partial y^2} = \frac{2}{25} \tan U (2 \tan^2 U - 3)$$

$$U = \sin^{-1}(x^2 + y^2)^{1/5}$$

$$\sin U = (x^2 + y^2)^{1/5}$$

$$= (x^2 + y^2)^{1/5}$$

$$= t^{2/5} F(U)$$

$F(U) = \sin U$ is a homoge. function
of degree $n = 2/5$

$$F(U) = \sin U$$

$$F'(U) = \cos U$$

By Cor. 3

$$x^2 \cdot \frac{\partial^2 U}{\partial x^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 U}{\partial y^2}$$

$$= f(U) [f'(U) - 1]$$

$$= \frac{f(U)}{f'(U)} [f'(U) - 1]$$

$$= \frac{2 \tan U}{5} \left[\frac{2 \sec^2 U}{5} - 1 \right]$$

$$= \frac{2}{5} \tan U \left[\frac{2}{5} + \frac{2 \tan^2 U}{5} - 1 \right]$$

$$= \frac{2}{5} \tan U \left[\frac{2 \tan^2 U}{5} - 3 \right]$$

$$= \frac{2}{25} \tan U [2 \tan^2 U - 3]$$

$$5 \text{ IF } U = \tan^{-1} \left(\frac{x^3 + y^3}{x - y} \right)$$

$$F(U) = \tan U = \frac{x^3 + y^3}{x - y}$$

$$F'(U) = \sec^2 U$$

Replacing x by $x + t$ and y by $y + t$.

$$F(x+t, y+t) = \frac{x^3 + y^3 + 3}{x+t - y+t}$$

$$= F(U)$$

$F(U) = \tan U$ is homoge. function of degree $n=2$.

By Cro 3.

$$\frac{x^2 \partial^2 U}{\partial x^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} + y^2 \frac{\partial^2 U}{\partial y^2}$$

$$= g(U) [g'(U) - 1]$$

$$= \frac{F(U)}{F'(U)} [g'(U) - 1]$$

$$= \sin U \cos U [2 \cos(2U) - 1 - 1]$$

$$= 2 \sin U \cos U \cdot \cos 2U$$

$$= 2 \cos(3U) \cdot \sin U$$

Hence,

$$x^2 \cdot \frac{\partial^2 U}{\partial x^2} + 2xy \cdot \frac{\partial^2 U}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 U}{\partial y^2} = 2 \cos 3U \cdot \sin U$$

6 IF $U = \frac{x^2 y^2 z^2}{x^2 + y^2 + z^2} \cos \left(\frac{xy + yz}{x^2 + y^2 + z^2} \right)$

show that $x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} + z \frac{\partial U}{\partial z} = \frac{4x^2 + y^2 + z^2}{x^2 + y^2 + z^2} U$

Replacing x by $x+t$, y by $y+t$ and z by $z+t$

$$f(U) = \frac{x^2 +^2 \cdot y^2 +^2 + z^2 +^2}{x^2 +^2 + y^2 +^2 + z^2 +^2} + \cos \left[\frac{x+(y+t) + (t+z)}{x^2 +^2 + y^2 +^2 + z^2 +^2} \right]$$

$$= +^4 \left(\frac{x^2 y^2 z^2}{x^2 + y^2 + z^2} \right) + \cos \left(\frac{xy + yz}{x^2 + y^2 + z^2} \right)$$

$$= +^4 U$$

U is a homo. function of degree $n=4$

By Cor.

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = n \cdot U$$

$$= 4U$$

Hence,

prove that $x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = 4U$

Brain Spot

* Task: 8 Partial Differentiation of Implicit Function of two variable

2 IF $x^3 + y^3 - 3axy = 0$ then Find $\frac{dy}{dx}$ at $(1, 1)$

$$\text{Let } F(x, y) = x^3 + y^3 - 3axy$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -\frac{F_x}{F_y} \\ &= \frac{-(3x^2 - 3ay)}{(3y^2 - 3ax)} \end{aligned}$$

$$= \frac{-x^2 + ay}{y^2 - ax}$$

$$\begin{aligned} \therefore \frac{dy}{dx} \Big|_{(1,1)} &= \frac{-1 + 0}{1 - 0} \\ &= -1 \end{aligned}$$

3 IF $x^5 + y^5 - 5a^3x^2 = 0$ then Find $\frac{d^2y}{dx^2}$

$$\text{Let } F(x, y) = x^5 + y^5 - 5a^3x^2$$

$$\therefore \frac{dy}{dx} = \frac{-F_x}{F_y} = - \frac{(5x^4 - 10xa^3)}{(5y^4)}$$

$$\therefore \frac{dy}{dx} = \frac{10xa^3 - 5x^4}{5y^4}$$

$$\therefore \frac{dy}{dx} = \frac{10xa^3 - 5x^4}{5y^4} = \frac{2xa^3 - x^4}{y^4}$$

$$\therefore \frac{dy}{dx} = \frac{2xa^3 - x^4}{y^4}$$

Brain Spot

4 IF $x \cdot e^y + \sin(xy) + y - \log 2 = 0$ then
Find $\frac{dy}{dx}$ at $(0, \log 2)$

Let $F(x, y) = xe^y + \sin(xy) + y - \log 2$

$$\therefore \frac{dy}{dx} = \frac{-F_x}{F_y} = - \frac{[e^y + y \cos(xy)]}{xe^y + \cos(xy) + 1}$$

$$\therefore \frac{dy}{dx} = \frac{-e^y - \cos(xy) \cdot y}{xe^y + \cos(xy) + 1}$$

$$\frac{dU}{dx} = \frac{-e^{-\log^2 2} + \log 2 (\cos 0)}{0 + 0 + 1}$$

$$\frac{dU}{dx} = -2 - \log 2$$

5 Find $\frac{dy}{dx}$ when $y^{x^y} = \sin x$

Brain Spot

$$y^{x^y} = \sin x$$

$$\therefore x^y \log y = \log \sin x$$

Let $F(x, y) = x^y \log y - \log \sin x$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{[y x^{y-1} \log y - \cot x]}{x^y \cdot \log y \log x + x^y \cdot \frac{1}{y}}$$

$$= \frac{\cot x + y x^{y-1} \log y}{x^y \left(\log y \log x + \frac{1}{y} \right)}$$

* Task 9: Tangent at Plane and normal line.

1 Find the equation of the tangent plane and normal line to the surface $xyz = 6$ at $(1, 2, 3)$

$$\text{Let, } F(x, y, z) = xyz - 6$$

$$(x_0, y_0, z_0) = (1, 2, 3)$$

$$F_x(1, 2, 3) = yz = 6$$

$$F_y(1, 2, 3) = xz = 3$$

$$F_z(1, 2, 3) = xy = 2$$

-> Tangent at plane,

$$(x - x_0)F_x + (y - y_0)F_y + (z - z_0)F_z = 0$$

$$\therefore (x - 1)6 + (y - 2)3 + (z - 3)2 = 0$$

$$\therefore 6x - 6 + 3y - 6 + 2z - 6 = 0$$

$$\therefore 6x + 3y + 2z = 18$$

-> Normal line,

$$\therefore \frac{x-1}{6} = \frac{y-2}{3} = \frac{z-3}{2}$$

2 Find the tangent plane to the surface $z = 1 - \frac{1}{10}(x^2 + 4y^2)$ at $(1, 1, \frac{1}{2})$

~~$F(x, y, z) = 1 - \frac{1}{10}(x^2 + 4y^2)$~~

~~$(x_0, y_0, z_0) = (1, 1, \frac{1}{2})$~~

~~$F_x(1, 1, \frac{1}{2}) = \frac{-2x}{10} = -\frac{1}{5}$~~

~~$F_y(1, 1, \frac{1}{2}) = \frac{-8y}{10} = -\frac{4}{5}$~~

~~$F_z(1, 1, \frac{1}{2}) = 0$~~

~~$z = 1 - \frac{1}{10}(x^2 + 4y^2)$~~

~~$\therefore 10z = 10 - x^2 - 4y^2$~~

$\therefore f(x, y, z) = x^2 + 4y^2 + 10z = 10$
 $(x_0, y_0, z_0) = (1, 1, \frac{1}{2})$

$F_x(1, 1, \frac{1}{2}) = 2x = 2$

$F_z(1, 1, \frac{1}{2}) = 10$

$F_y(1, 1, \frac{1}{2}) = 8y = 8$

Brain Spot

→ Tangent at Plane,

$$C(x-1)2 + (y-1)8 + (z-\frac{1}{2})10 = 0$$

$$2x - 2 + 8y - 8 + 2z - \frac{10}{2} = 0$$

$$2x + 8y + 10z = 8 + 15$$

$$x + 4y + 5z = 10$$

→ Normal line

$$\therefore \frac{C(x-1)}{2} = \frac{y-1}{8} = \frac{z-\frac{1}{2}}{10}$$

3 Find the equation of the tangent plane and normal line to the surface $z + 8 = x e^y \cos z$ at point $(8, 0, 0)$

$$\therefore x e^y \cos z - z = 8$$

$$\text{Let } F(x, y, z) = x e^y \cos z - z - 8 = 0$$

$$(x_0, y_0, z_0) = (8, 0, 0)$$

$$\Rightarrow F_x = e^y \cos z = 1$$

$$F_y = x e^y \cos z = 8$$

$$F_z = \sin z - 1 = -1$$

=> Tangent at Plane,

$$(x-8)1 + (y-0)8 + (-1)(z) = 0$$

$$\therefore x-8 + 8y - z = 0$$

$$\therefore x + 8y - z = 8$$

=> Normal line is,

$$\therefore \frac{x-8}{1} = \frac{y-0}{8} = \frac{z}{-1}$$

$$\therefore x-8 = \frac{y}{8} = \frac{z}{-1}$$

Brain Spot

4 Show that plane $3x + 12y - 6z - 17 = 0$ touches the surface $3x^2 - 6y^2 + 9z^2 + 17 = 0$ Find point.

Point = $P(x_1, y_1, z_1)$

$$f(x, y, z) = 3x^2 - 6y^2 + 9z^2 + 17$$

$$\frac{\partial f}{\partial x} = 6x = 6x_1$$

$$f_y = -12y = -12y_1$$

$$f_z = -18z = -18z_1$$

=> Tangent at plan

$$\therefore (x - x_1)(6x_1) + (y - y_1)(-12y_1) + (z - z_1)(18z_1) = 0$$

$$\therefore 6xx_1 - 6x_1^2 + 12yy_1 + 12x_1^2 + 18zz_1 - 18z_1^2 = 0$$

$$\therefore 3xx_1 - 6yy_1 + 9zz_1 + 17 = 0$$

=> Comparing with $3x + 12y - 6z - 17 = 0$

$$\therefore \frac{x_1}{1} = \frac{y_1}{2} = \frac{z_1}{-2} = -1$$

$$\therefore P(x_1, y_1, z_1) = (-1, 2, 2/3)$$

Task: 10: Jacobin

1 Find the Jacobian $J \begin{pmatrix} U, V \\ x, y \end{pmatrix}$ for

$$U = e^x \sin y, \quad V = x \log(\sin y)$$

$$\Rightarrow J = \frac{\partial(U, V)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} e^x \sin y & e^x \cos y \\ \log \sin y & \frac{x \cos y}{\sin y} \end{vmatrix}$$

$$= \begin{vmatrix} e^x \sin y & e^x \cos y \\ \log(\sin y) & x \cot y \end{vmatrix}$$

$$= (e^x \cdot x \sin y \cdot \cot y - e^x \cos y \cdot \log(\sin y))$$

$$= e^x \cdot x \cos y - e^x \cos y \cdot \log \sin y$$

$$J = \frac{\partial(U, V)}{\partial(x, y)} = e^x \cos y (x - \log(\sin y))$$

2 Calculate $\frac{\partial(x, y)}{\partial(u, v)}$ for $x = e^u \cos v$
 $y = e^u \sin v$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$=$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}$$

$$= e^{2u} (\cos^2 v + \sin^2 v)$$

$$\frac{\partial(x, y)}{\partial(u, v)} = e^{2u} = J^*$$

$$J * J^* = 1$$

$$J = \frac{1}{J^*} = \frac{1}{e^{2u}} = e^{-2u} = \frac{\partial(u, v)}{\partial(x, y)}$$

3 If $U = xyz$, $V = x^2 + y^2 + z^2$, $W = x + y + z$
then Find $\frac{\partial(U, V, W)}{\partial(x, y, z)}$

$$J^* = \frac{\partial(U, V, W)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \frac{\partial W}{\partial z} \end{vmatrix}$$

Brain Spot

$$= \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= yz(2y - 2z) - xz(2x - 2z) + xy(2x - 2y)$$

$$= 2y^2z - 2z^2y - 2x^2z + 2z^2x$$

$$+ 2x^2y - 2y^2x$$

$$= 2(y^2z - z^2y - x^2z + z^2x + x^2y - xy^2)$$

$$= 2(zy^2 - z^2y - x^2z + z^2x + x^2y - xy^2)$$

$$= -2(x - y)(y - x)(z - x)$$

4. IF $x = r \sin \theta \cdot \cos \phi$, $y = r \sin \theta \sin \phi$
 $z = r \cos \theta$ then show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta$$

$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$	$\frac{\partial x}{\partial r}$	$\frac{\partial x}{\partial \theta}$	$\frac{\partial x}{\partial \phi}$
	$\frac{\partial y}{\partial r}$	$\frac{\partial y}{\partial \theta}$	$\frac{\partial y}{\partial \phi}$
	$\frac{\partial z}{\partial r}$	$\frac{\partial z}{\partial \theta}$	$\frac{\partial z}{\partial \phi}$
	$\frac{\partial z}{\partial r}$	$\frac{\partial z}{\partial \theta}$	$\frac{\partial z}{\partial \phi}$

Brain Spot

$\sin \theta \cdot \cos \phi$	$r \cos \theta \cdot \cos \phi$	$-r \sin \theta \sin \phi$
$\sin \theta \cdot \sin \phi$	$r \cos \theta \cdot \sin \phi$	$r \cos \theta \cdot \sin \phi$
$\cos \theta$	$-r \sin \theta$	0

$$= \sin \theta \cdot \cos \phi (r^2 \sin^2 \theta \cdot \cos \phi) -$$

$$r \cos \theta \cdot \cos \phi (-r \cos \theta \cdot \cos \phi \sin \theta)$$

$$+ (-r \sin \theta \cdot \sin \phi) \left(\frac{-r \sin^2 \theta}{\sin \phi} - \frac{r \cos^2 \theta}{\sin \phi} \right)$$

$$= r^2 \sin^3 \theta \cdot \cos^2 \phi + r^2 \cos^2 \theta \cos^2 \phi \sin \theta$$

$$+ (r^2 \sin^3 \theta \sin^2 \phi + r^2 \cos^2 \theta \sin^2 \phi)$$

$$= r^2 \sin \theta$$

5 IF $x = U$, $y = U \tan v$, $z = w$ then
verify the $JJ' = 1$

$$\Rightarrow J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$J = \begin{vmatrix} 1 & 0 & 0 \\ \tan v & U \sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$J = 1(U \sec^2 v) = U \sec^2 v \quad \text{--- (1)}$$

$$\Rightarrow J^* = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$J^* = \begin{vmatrix} 1 & 0 & 0 \\ x^2 - y^2 & x & 0 \\ x^2 + y^2 & x^2 + y^2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= U$$

$$U^2(1 + \tan^2 v)$$

$$= \frac{1}{U \sec^2 v} \quad \text{--- (2)}$$

$$\Rightarrow \text{L.H.S.} = J^* J^*$$

$$= U \sec^2 v \cdot \frac{1}{U \sec^2 v}$$

$$= 1 = J^* J^* \text{ is prove.}$$

* Task 11: Taylor's Theorem for the Function of two variable

1 Expand $F(x, y) = x^2y + 3y - 2$ in power of $(x-1)$ and $(y+2)$

$$\text{Let } F(x, y) = x^2y + 3y - 2 = (1)^2(-2)^2 + 3(-2) - 2$$

$$F(a, b) = -10$$

=> Taylor's Series,

$$F(x, y) = F(a, b) + [(x-a)F_x(a, b) + (y-b)F_y(a, b) + \frac{1}{2}[(x-a)^2F_{xx}(a, b) + 2(x-a)(y-b)F_{xy}(a, b) + (y-b)^2F_{yy}(a, b)] + \dots$$

=> Here, $x=1$, $y=-2$

$$\Rightarrow F_x = 2xy = 2(-2) = -4$$

$$\Rightarrow F_y = x^2 + 3 = 4$$

$$\Rightarrow F_{xx} = 2$$

$$\Rightarrow F_{xy} = 2x = 2$$

$$\Rightarrow F_{yy} = 0$$

$$\Rightarrow F(x, y) = -10 + [(x-1)(-4) + (y+2)4] +$$

$$1 [(x-1)^2(2) + 2(x-1)(y+2) + 0]$$

2 Expand $e^x \cos y$ in power of x and $(y - \frac{\pi}{4})$ upto second degree.

$$\Rightarrow \text{Let } F(x, y) = e^x \cos y$$

$$\Rightarrow \text{Here } a=0 \text{ and } b = \frac{\pi}{4}$$

\Rightarrow Taylor's Series,

$$F(x, y) = F(a, b) + (x-a)F_x + (y-b)F_y$$

$$+ \frac{1}{2!} \left[(x-a)^2 F_{xx} + 2(x-a)(y-b)F_{xy} + (y-b)^2 F_{yy} \right] + \dots$$

$$\rightarrow F_x = e^x \cos y = \frac{1}{\sqrt{2}}$$

$$\rightarrow F_y = -e^x \sin y = -\frac{1}{\sqrt{2}}$$

$$\rightarrow F_{xx} = e^x \cos y = \frac{1}{\sqrt{2}}$$

$$\rightarrow F_{yy} = -e^x \cos y = -\frac{1}{\sqrt{2}}$$

$$\rightarrow F_{xy} = -e^x \sin y = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow e^{x \cos y} = \frac{1}{\sqrt{2}} + \left(\frac{x - (y - \frac{\pi}{4})}{\sqrt{2}} \right) + \frac{1}{2} \left(\frac{x^2 - 2x(y - \frac{\pi}{4}) - (y - \frac{\pi}{4})^2}{\sqrt{2}} \right) + \dots$$

3 Expand $\cos x \cos y$ in Power of x and y upto third degree terms.

Here $a = 0$ and $b = 0$

\Rightarrow Taylor's Serie,

$$F(x, y) = F(a, b) + \left[(x) F_x + y F_y \right] + \frac{1}{2!} \left[x^2 F_{xx} + 2xy F_{xy} + y^2 F_{yy} \right] + \frac{1}{3!} \left[x^3 F_{xxx} + 3x^2 y F_{x^2 y} + 3x^2 x F_{y^2 x} + y^3 F_{yyy} \right] + \dots$$

$$\rightarrow F_x = -\sin x \cos y = 0$$

$$\rightarrow F_y = -\cos x \sin y = 0$$

$$\rightarrow F_{xx} = -\cos x \cos y = -1$$

$$\rightarrow F_{xxx} = \sin x \cos y = 0$$

- > $F_{yy} = -\cos x \cos y = -1$
- > $F_{yyy} = +\cos x \sin y = 0$
- > $F_{x^2y} = \cos x \sin y = 0$
- > $F_{y^2x} = \sin x \cos y = 0$

$$\Rightarrow \cos x \cos y = 1 + [x(0) + y(0)] + \frac{1}{2!} [x^2(-1) + 2xy + y^2(-1)] + \frac{1}{3!} (0) + \dots$$

$$\cos x \cos y = 1 - \frac{x^2}{2} - \frac{y^2}{2} + \dots$$

4 Expand $\tan^{-1}(y/x)$ about (1,1).

Here $a=1, b=1$

$$\rightarrow F(a, b) = \frac{\pi}{4}$$

$$\rightarrow F_x = \frac{1}{1 + y^2/x^2} \cdot \left(\frac{-y}{x^2} \right) = \frac{-1}{2}$$

$$\rightarrow F_{xx} = \frac{-y(2xc)}{(x^2 + y^2)^2} = \frac{1}{2}$$

$$\rightarrow F_y = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} = \frac{1}{2}$$

$$\rightarrow F_{yy} = \frac{1 - 2x(2y)}{(x^2 + y^2)^2} = \frac{-1}{2}$$

$$\rightarrow F_{xy} = \frac{-[2x^2 + y^2 - 2y^2]}{(x^2 + y^2)^2} = 0$$

$$\rightarrow F(x, y) = \frac{\pi}{4} + \left[(x-1)(-1) + (y-1)\left(\frac{1}{2}\right) \right]$$

$$+ \frac{1}{2!} \left[\frac{(x-1)^2}{4} + \frac{-(y-1)^2}{4} \right] + \dots$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2$$

$$- \frac{1}{4}(y-1)^2 + \dots$$

* Task 12: Maxima, Minima For the Function of two variables.

1 Find the extreme values of $x^2 + y^2 + xy + x - 4y + 5$.

$$\Rightarrow \text{Let } F(x, y) = x^2 + y^2 + xy + x - 4y + 5$$

$$\frac{\partial F}{\partial x} = 2x + y + 1 = 0$$

$$\frac{\partial F}{\partial y} = 2y + x - 4 = 0$$

$$\therefore 2y = 4 - x$$

$$\therefore y = 4 - \frac{x}{2}$$

$$\therefore 2x + \frac{4 - x}{2} + 1 = 0$$

$$\therefore 4x + 4 - x + 2 = 0$$

$$\therefore 3x + 6 = 0$$

$$\therefore x = -2 \rightarrow y = \frac{4 + 2}{2} = 3$$

point is $(-2, 3)$

$$\Rightarrow r = \frac{\partial^2 F}{\partial x^2} = 2, \quad t = \frac{\partial^2 F}{\partial y^2} = 2$$

$$S = \frac{\partial^2 F}{\partial x \partial y} = 1$$

$$\Rightarrow x + -S^2 = 4 - 1 = 3 > 0$$

$$y = 2 > 0$$

Hence, $F(x, y)$ is minimum at $(-2, 3)$ point

$$F_{\min} = (-2)^2 + (-2)^2 + (-2)(3) - 2 - 4(3) + 5$$

$$= 4 - 6 - 2 - 12 + 5$$

$$F_{\min} = -2$$

2 Find extreme values of $x^3 + 3xy^2 - 15y^2 + 72x$

$$\Rightarrow \text{Let } f(x, y) = x^3 + 3xy^2 - 15x^2 + 72x$$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 + 72 = 0$$

$$+ 30x^2$$

$$\therefore x^2 + y^2 + 24 = 0$$

$$\frac{\partial f}{\partial y} = 6xy - 30y = 0$$

$$\therefore xy - 5y = 0$$

$$\therefore y(x - 5) = 0$$

$$\Rightarrow x = 5, y = 0$$

$$\rightarrow \text{For } x = 5, \rightarrow 3(5)^2 + 3(y^2) + 72 + 3(5y) = 0$$

$$\therefore y = \pm 2$$

$$\rightarrow \text{For } y = 0$$

$$\therefore 3x^2 + 3y^2 - 30x - 72 = 0$$

$$\therefore 3x^2 - 30x + 72 = 0$$

$$\therefore x = 6, x = 4$$

$$\rightarrow \text{points are } (6, 0), (4, 0), (5, +2), (5, -2)$$

$$\Rightarrow r = \frac{\partial^2 f}{\partial x^2} = 6x - 30$$

$$\Rightarrow s = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$\Rightarrow t = \frac{\partial^2 f}{\partial y^2} = 6x - 30$$

$$\Rightarrow r + t - s^2 = (6x - 30)(6x - 30) - 36y^2$$

=)	(x, y)	r	s	t	$rt - s^2$	Conclusion
1	$(4, 0)$	$-6 < 0$	0	-6	$36 > 0$	maximum
2	$(6, 0)$	$6 > 0$	0	6	$36 > 0$	minimum
3	$(5, 2)$	0	6	0	$-36 < 0$	neither max. or min.
4	$(5, 2)$	0	-6	0	$-36 < 0$	neither max or min

Hence, $F(x, y)$ is maximum at $(4, 0)$ and minimum at $(6, 0)$

$$F_{\max} = 112$$

$$F_{\min} = 108$$

3 Find the extreme values of
 $f(x, y) = \sin x + \sin y + \sin(x+y)$

$$\Rightarrow \frac{\partial f}{\partial x} = \cos x + \cos(x+y) = 0$$

$$\Rightarrow \frac{\partial f}{\partial y} = \cos y + \cos(x+y) = 0$$

$$\therefore \cos x = \cos y$$

$$\therefore x = y$$

$$\Rightarrow \cos x + \cos 2x = 0$$

$$\therefore \cos x = -\cos 2x$$

$$\therefore x = \pi - 2x$$

$$\therefore 3x = \pi$$

$$\Rightarrow \cos x = \cos(\pi + 2x)$$

$$\therefore x = \pi + 2x$$

$$\therefore x = -\pi$$

$$\Rightarrow \cos y + \cos 2y = 0$$

$$\therefore y = \pi/3, -\pi$$

$$\Rightarrow \text{point } \left(\frac{\pi}{3}, \frac{\pi}{3}\right), (-\pi, -\pi)$$

$$\Rightarrow r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$$

$$\Rightarrow s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$$

$$\Rightarrow t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$$

	(x, y)	r	s	t	$r + -s^2$	Conclusion
1	$(\frac{\pi}{3}, \frac{\pi}{3})$	$-\sqrt{3} < 0$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$9/4 > 0$	maximum
2	$(-\pi, -\pi)$	0	0	0	0	no conclusion

$$\Rightarrow F_{\max} = \frac{3\sqrt{3}}{2}$$

5 Determine the points where the function $x^3 + y^3 - 3axy$ has maximum or minimum.

$$\Rightarrow \text{Let, } F(x, y) = x^3 + y^3 - 3axy$$

$$\Rightarrow \frac{\partial F}{\partial x} = 3x^2 - 3ay = 0 = x^2 - ay$$

$$\frac{\partial F}{\partial y} = 3y^2 - 3ax = 0 = y^2 - ax$$

$$\therefore y^2 = ax$$

$$\therefore \text{putting } y = \frac{x^2}{a}$$

$$\therefore x^4 - a^3x = 0$$

$$\therefore x(x-a)(x^2+ax+a^2) = 0$$

$$\therefore x = 0 ; x = a$$

$$y = 0 ; y = a$$

\Rightarrow point $(0,0)$ and (a,a)

$$\Rightarrow r = \frac{\partial^2 F}{\partial x^2} = 2x$$

$$\Rightarrow s = \frac{\partial^2 F}{\partial x \partial y} = -a$$

$$\Rightarrow t = \frac{\partial^2 F}{\partial y^2} = 2y$$

\Rightarrow For (a, a) point

$$r > 0 \quad 3a^2 > 0$$

Hence, $F(x, y)$ is minimum at (a, a)

$$F_{\min} = -a^3$$

- 4 A rectangular box open at the top is to have a volume 108 cubic meter. What must be the dimensions so that the total surface area is minimum